Selected Solutions to Walter Rudin's Principles of Mathematical Analysis Third Edition

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Contents

Chapter 1

The Real and Complex Number Systems

Unless explicitly stated otherwise, all numbers in this chapter's exercises are understood to be real.

1.1 Exercise 1

If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. Suppose r is rational, so that $r = a/b$ for $a, b \in Z$, and let x be irrational. Now assume for contradiction that $r + x$ is rational. Then, for some integers c and d, we can write

$$x = (r + x) - r = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd},$$

so that x is rational: a contradiction. Therefore $r + x$ must be irrational.

Similarly, if we assume that $rx = c/d$ for $c, d \in Z$, then, since $r \neq 0$,

$$x = \frac{1}{r}(rx) = \frac{b}{a} \cdot \frac{c}{d} = \frac{bc}{ad},$$

again a contradiction. Hence rx must be irrational. □

1.2 Exercise 2

Prove that there is no rational number whose square is 12.

Proof. Assume the contrary, and let p and q be integers such that $p^2 = 12q^2$. We may further suppose that p and q have no common factors other than 1. Then 2 divides p^2 so that 2 divides p as well. Thus we can write $p = 2k$ for $k \in Z$. Then $4k^2 = 12q^2$, which implies that $k^2 = 3q^2$.

Now 3 divides k^2, so 3 divides k as well (if 3 does not divide k, then 3 could not divide k^2 since 3 is prime), allowing us to write $k = 3\ell$ for $\ell \in Z$. Hence $3q^2 = 9\ell^2$ which implies that $q^2 = 3\ell^2$. Then 3 divides q^2 and thus q as well.

Since 3 divides k, it must divide p, and we see that 3 is a common factor of p and q, which contradicts our choice of p and q. Therefore there is no rational number p/q whose square is 12. □

1.3 Exercise 3

Prove Proposition 1.15:

Proposition. *Let F be a field and let $x, y, z \in F$. Then the following properties hold.*

(a) *If $x \neq 0$ and $xy = xz$ then $y = z$.*

(b) *If $x \neq 0$ and $xy = x$ then $y = 1$.*

(c) *If $x \neq 0$ and $xy = 1$, then $y = 1/x$.*

(d) *If $x \neq 0$ then $1/(1/x) = x$.*

Proof. (a) Let x, y, z be such that $xy = xz$, with $x \neq 0$. Since x is nonzero, it has a multiplicative inverse $1/x$. So

$$y = 1y = \left(\frac{1}{x} \cdot x\right) y = \frac{1}{x}(xy) = \frac{1}{x}(xz) = \left(\frac{1}{x} \cdot x\right) z = 1z = z.$$

(b) Suppose $xy = x$, $x \neq 0$. As before, $1/x$ exists, so we have

$$y = 1y = \left(\frac{1}{x} \cdot x\right) y = \frac{1}{x}(xy) = \frac{1}{x} \cdot x = 1.$$

(c) If $xy = 1$, $x \neq 0$, then

$$y = 1y = \left(\frac{1}{x} \cdot x\right) y = \frac{1}{x}(xy) = \frac{1}{x} \cdot 1 = \frac{1}{x}.$$

(d) Let $x \neq 0$. Since $(1/x)x = 1$, it follows from part (c) above that $x = 1/(1/x)$.

□

1.4 Exercise 4

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since E is nonempty, we may choose $n \in E$. α is a lower bound of E, so $\alpha \leq n$. On the other hand, β is an upper bound, so $n \leq \beta$. By the transitive property for ordered sets (see Definition 1.5 in the text), $\alpha \leq n$ and $n \leq \beta$ together imply that $\alpha \leq \beta$. □

1.5 Exercise 5

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. Let $a = \inf A$ and choose any $b \in -A$ (we know $-A$ is nonempty since A is nonempty). Then $-b \in A$ so that $a \leq -b$, which implies $b \leq -a$. And b was arbitrary, so $-a = -\inf A$ is an upper bound for $-A$.

$-A$ is bounded above, so it has a least upper bound $c = \sup(-A)$. We need to show that $c = -a$. Assume the contrary, and suppose $c < -a$. Then since c is an upper bound for $-A$, we have $c \geq b$ for all b in $-A$. Then $-c \leq -b$, with $-b \in A$, so that $-c$ is a lower bound for A. But $c < -a$, so $-c > a$. Hence $-c$ is a lower bound for A, and it is larger than $a = \inf A$, a contradiction. This shows that $c = -a$, so that $\sup(-A) = -\inf A$ as required. □

1.8 Exercise 8

Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. Assume the contrary, so that the set of complex numbers is an ordered field with order $<$. Then by Proposition 1.18d, $-1 = i^2 > 0$. But then $0 = -1 + 1 > 0 + 1 = 1$. Hence $0 > 1$. But again by Proposition 1.18d, $1 = 1^2 > 0$. This is a contradiction, so the complex numbers cannot be an ordered field. □

1.9 Exercise 9

Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. Does this ordered set have the least-upper-bound property?

Note. For clarity, we will use the symbol $\lessdot$ to represent the complex ordering defined above. For the rest of this exercise, the ordinary $<$ symbol will only denote the usual ordering on R.

Proof. Let $z = a + bi$ and $w = c + di$ be arbitrary complex numbers, with $a, b, c, d \in R$. Since R is an ordered field, exactly one of the statements $a < c$, $a > c$, or $a = c$ must be true. We consider each case in turn: First, if $a < c$ then $z \lessdot w$ but $w \not\lessdot z$ and certainly $z \neq w$. Next, if $a > c$, then $w \lessdot z$ while $z \not\lessdot w$ and $z \neq w$.

In the case where $a = c$, then either $b < d$, $b > d$, or $b = d$. If $b < d$ then $z \lessdot w$ while $w \not\lessdot z$ and $z \neq w$. Similarly, if $b > d$, then $w \lessdot z$ while $z \not\lessdot w$ and $z \neq w$. And if $b = d$, then $z = w$ and neither of the statements $z \lessdot w$ and $w \lessdot z$ are true.

In every case, exactly one of $z \lessdot w$, $w \lessdot z$, or $z = w$ is true.

Lastly, suppose that $x = a_1 + b_1 i$, $y = a_2 + b_2 i$, and $z = a_3 + b_3 i$ are complex numbers with $a_k, b_k \in R$ and such that $x \lessdot y$ and $y \lessdot z$. There are four cases: If $a_1 < a_2 < a_3$, or if $a_1 < a_2 = a_3$, or if $a_1 = a_2 < a_3$ then $a_1 < a_3$ and $x \lessdot z$.

The last case is where $a_1 = a_2 = a_3$. In that case we must have $b_1 < b_2 < b_3$ so that $b_1 < b_3$ and $x \lessdot z$. This completes the proof. □

Claim. *This ordered set does not have the least-upper-bound property.*

Proof. Define A to be the set of all complex numbers $a + bi$ with $a, b \in R$ such that $a < 0$. Then clearly A is bounded above by 0. Now suppose that $z = x + yi$ is any upper bound for A.

First note that $x \geq 0$. For, if not, we could choose a real number x' such that $x < x' < 0$. But then $x' \in A$ and $z \lessdot x'$, which would give a contradiction.

Now let y' be any real number less than y. Since $x \geq 0$, the complex number $x + y'i$ is an upper bound for A. But $x + y'i \lessdot z$, so z cannot be the least upper bound for A. Since z was arbitrary, this shows that the nonempty set A, which is bounded above, has no least upper bound. □

1.10 Exercise 10

Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}. \tag{1.1}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof. Direct computation gives

$$\begin{aligned}
z^2 &= a^2 - b^2 + 2abi \\
&= \frac{|w| + u}{2} - \frac{|w| - u}{2} + 2\left(\frac{|w| + u}{2}\right)^{1/2}\left(\frac{|w| - u}{2}\right)^{1/2} i \\
&= u + \left(|w|^2 - u^2\right)^{1/2} i \\
&= u + \left(v^2\right)^{1/2} i \\
&= u + |v| i.
\end{aligned}$$

Therefore $z^2 = w$ if $v \geq 0$ and $z^2 = \overline{w}$ if $v \leq 0$. In the latter case, we have $(\bar{z})^2 = \overline{(z^2)} = w$ (by Theorem 1.31b).

Now, let $w = u + vi$ be any complex number, and define a and b as in (1.1). If $v > 0$, then $z = a + bi$ and $-z = -a - bi$ are distinct complex numbers such that $z^2 = w$ and $(-z)^2 = w$. On the other hand, if $v < 0$, then $\bar{z} = a - bi$ and $-\bar{z} = -a + bi$ are distinct values with $(\bar{z})^2 = w$ and $(-\bar{z})^2 = w$. And lastly, if $v = 0$ and $u \neq 0$ then $y = |u|^{1/2}$ and $-y = -|u|^{1/2}$ are distinct with $y^2 = w$ and $(-y)^2 = w$. Therefore the only complex number that does not have two distinct complex square roots is 0 itself. □

1.11 Exercise 11

If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Proof. If $z = 0$, we may take $r = 0$ and $w = 1$.

Otherwise, $|z| > 0$ and we may simply let

$$r = |z| \quad \text{and} \quad w = \frac{z}{|z|}.$$

Then $z = rw$, and $r \geq 0$ by Theorem 1.33a. Moreover,

$$|w| = (w\overline{w})^{1/2} = \left(\frac{z\bar{z}}{|z|^2}\right)^{1/2} = 1.$$

□

Claim. *w and r are uniquely determined by z if and only if $z \neq 0$.*

Proof. First, fix a nonzero complex number z. Suppose $z = r_1 w_1 = r_2 w_2$, where $r_1, r_2 \geq 0$ and $|w_1| = |w_2| = 1$. Then

$$r_1 = |r_1||w_1| = |z| = |r_2||w_2| = r_2$$

and, since z is nonzero, r_1 and r_2 are positive and we have

$$w_1 = \frac{r_2 w_2}{r_1} = w_2.$$

Therefore w and r are uniquely determined.

To prove the other direction, suppose $z = 0$. Then we may let $r = 0$, $w_1 = 1$, and $w_2 = -1$ so that $z = rw_1 = rw_2$ but $w_1 \neq w_2$. □

1.12 Exercise 12

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|. \tag{1.2}$$

Proof. We use induction on n. The case where $n = 1$ is trivial. Now suppose (1.2) holds for $n = k$ where k is any positive integer. Then by Theorem 1.33e and the induction hypothesis we have

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_k + z_{k+1}| &= |z_1 + z_2 + \cdots + z_{k-1} + (z_k + z_{k+1})| \\
&\leq |z_1| + |z_2| + \cdots + |z_{k-1}| + |z_k + z_{k+1}| \\
&\leq |z_1| + |z_2| + \cdots + |z_{k-1}| + |z_k| + |z_{k+1}|.
\end{aligned}$$

Therefore (1.2) holds for all positive integers n. □

1.13 Exercise 13

If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|.$$

Proof. The triangle inequality from Theorem 1.33e gives

$$|x| = |x - y + y| \leq |x - y| + |y|$$

so $|x| - |y| \leq |x - y|$. Similarly,

$$|y| = |y - x + x| \leq |x - y| + |x|$$

which gives $|y| - |x| \leq |x - y|$.

Now there are two cases:

$$\big||x| - |y|\big| = |x| - |y| \quad \text{if } |x| - |y| \geq 0,$$

or

$$\big||x| - |y|\big| = |y| - |x| \quad \text{if } |x| - |y| \leq 0.$$

Either way, we get $\big||x| - |y|\big| \leq |x - y|$. □

1.14 Exercise 14

If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Solution. Using Theorem 1.31, we get

$$\begin{aligned} |1 + z|^2 &= (1 + z)\overline{(1 + z)} \\ &= (1 + z)(1 + \bar{z}) \\ &= 1 + z + \bar{z} + z\bar{z} \\ &= 2 + 2\operatorname{Re} z, \end{aligned}$$

and, similarly,

$$\begin{aligned} |1 - z|^2 &= (1 - z)(1 - \bar{z}) \\ &= 1 - z - \bar{z} + z\bar{z} \\ &= 2 - 2\operatorname{Re} z. \end{aligned}$$

Hence

$$|1 + z|^2 + |1 - z|^2 = 4.$$

□

1.15 Exercise 15

Under what conditions does equality hold in the Schwartz inequality?

Solution. Let A, B, and C be defined as in the proof of Theorem 1.35. From that proof, we have equality when $Ba_j = Cb_j$ for each j from 1 to n. This will be the case when $a_j = kb_j$ for some constant $k = C/B$. □

1.17 Exercise 17

Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. For any $\mathbf{x}, \mathbf{y} \in R^k$, we have

$$\begin{aligned}
|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= \left(|\mathbf{x}|^2 + 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2\right) + \left(|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2\right) \\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2. \qquad \square
\end{aligned}$$

Geometrically, this result says that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on its sides. When $\mathbf{x}\cdot\mathbf{y} = 0$ the parallelogram is a rectangle and the statement reduces to the Pythagorean Theorem.

1.18 Exercise 18

If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x}\cdot\mathbf{y} = 0$. Is this also true if $k = 1$?

Proof. Let $\mathbf{x} = (x_1, \ldots, x_k)$. If x_1 or x_2 are nonzero, then we may let $\mathbf{y} = (x_2, -x_1, 0, 0, \ldots, 0)$ (provided $k > 1$). Then $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x}\cdot\mathbf{y} = 0$. On the other hand, if either or both of x_1, x_2 are 0, then let $\mathbf{y}$ be the vector whose coordinates are all 0 except with a 1 in the same position as one of the zero coordinates from $\mathbf{x}$. Again, $\mathbf{y}$ is nonzero while $\mathbf{x}\cdot\mathbf{y} = 0$.

Finally, we note that the result is *not* true for $k = 1$. For example, if $x = 1$, then there is no nonzero y with $xy = 0$. $\square$

1.19 Exercise 19

Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}| \tag{1.3}$$

if and only if $|\mathbf{x}-\mathbf{c}| = r$.

Solution. Set

$$\mathbf{c} = \frac{1}{3}(4\mathbf{b}-\mathbf{a}) \quad \text{and} \quad r = \frac{2}{3}|\mathbf{b}-\mathbf{a}|.$$

Since norms are nonnegative, $|\mathbf{x}-\mathbf{c}| = r$ if and only if

$$|\mathbf{x}-\mathbf{c}|^2 = r^2,$$

or

$$\left|\mathbf{x} - \frac{1}{3}(4\mathbf{b}-\mathbf{a})\right|^2 = \frac{4}{9}|\mathbf{b}-\mathbf{a}|^2.$$

This becomes

$$\left(\mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\right) \cdot \left(\mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\right) = \frac{4}{9}(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}).$$

Expanding and simplifying then reduces this equation to

$$|\mathbf{x}|^2 - \frac{8}{3}\mathbf{b} \cdot \mathbf{x} + \frac{2}{3}\mathbf{a} \cdot \mathbf{x} + \frac{4}{3}|\mathbf{b}|^2 - \frac{1}{3}|\mathbf{a}|^2 = 0.$$

Finally, if we square and expand equation (1.3) in the same way, we see that it reduces to the same equation above. Therefore (1.3) holds if and only if $|\mathbf{x} - \mathbf{c}| = r$. □

1.20 Exercise 20

With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Solution. The proof that the set has the least-upper-bound property is the same as the proof given in Step 3 in the Appendix, except that we don't need the part which shows that (III) holds.

Similarly, the proofs for (A1) through (A3) do not require modification. However, the proof for (A4) doesn't quite work since it makes use of property (III).

Instead, define

$$0' = \{r \in Q \mid r \leq 0\}.$$

Then for any cut α, if $r \in \alpha$ and $s \in 0'$ then $r + s \leq r$ so $r + s \in \alpha$. Therefore $\alpha + 0' \subset \alpha$. Now pick $p \in \alpha$. Then $p + 0 \in \alpha + 0'$ so that $\alpha \subset \alpha + 0'$. This shows (A4).

Finally, define $0^* = \{r \in Q \mid r < 0\}$. Then 0^* is a cut since it satisfies properties (I) and (II), however there is no cut α such that $0^* + \alpha = 0'$. If there is such an α, then in particular $0 \in 0^* + \alpha$ so that for some $r \in 0^*$, we have $-r \in \alpha$. But $r < 0$ so $r/2 \in 0^*$. Then $r/2 + (-r) \in 0^* + \alpha$ but $r/2 - r > 0$ which shows $r/2 - r \notin 0'$. Hence $0^* + \alpha \neq 0'$ so (A5) does not hold. □

Chapter 2

Basic Topology

2.1 Exercise 1

Prove that the empty set is a subset of every set.

Proof. Let A be any set. Since the empty set has no elements, it is vacuously true that for every x in the empty set, $x \in A$. $\square$

2.2 Exercise 2

A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0. \tag{2.1}$$

Prove that the set of all algebraic numbers is countable.

Proof. For each positive integer N, let E_N denote the set of all algebraic numbers z satisfying (2.1) where

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Since all the terms on the left-hand side are positive integers, it follows that for each N there are only finitely many such polynomial equations. And for any fixed $n \leq N$, any polynomial of degree n has a finite number of roots. Hence the set E_N is finite.

If A denotes the set of all algebraic numbers, then we have

$$A = \bigcup_{N=1}^{\infty} E_N.$$

Being the union of countably many finite sets, A must be at most countable by the corollary to Theorem 2.12. And since A must be infinite (for example, Z is a subset) this shows that A is countable. $\square$

2.3 Exercise 3

Prove that there exist real numbers which are not algebraic.

Proof. By the previous exercise, we know that the set A of algebraic numbers is countable. If $A = R$, then R is countable, so A must be a proper subset of R. Thus we can find an $x \in R$ with $x \notin A$. □

2.4 Exercise 4

Is the set of all irrational real numbers countable?

Solution. Suppose that the irrationals $R - Q$ are countable. Since

$$R = Q \cup (R - Q),$$

this means that R is the union of countable sets and is therefore countable by Theorem 2.12. This contradiction shows that $R - Q$ is uncountable. □

2.5 Exercise 5

Construct a bounded set of real numbers with exactly three limit points.

Solution. For each integer m, consider the set

$$E_m = \left\{ m + \frac{1}{n+1} \;\middle|\; n \in Z^+ \right\},$$

where Z^+ denotes the positive integers. Then the set $A = E_0 \cup E_1 \cup E_2$ has exactly three limit points, namely the points 0, 1, and 2, which we will now demonstrate. First note that any neighborhood of 0 must contain points in E_0, and similarly for 1 and 2, so that $0, 1, 2 \in A'$.

On the other hand, suppose $x \in R - \{0, 1, 2\}$. Let

$$r = \min\{|x|, |1 - x|, |2 - x|, |3 - x|\}.$$

Then the interval $(x - r/2, x + r/2)$ contains finitely many points in A. So by Theorem 2.20, it follows that x is not a limit point of A.

Therefore, the only limit points of A are 0, 1, and 2. □

2.6 Exercise 6

Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Proof. First we will show that any limit point of $\overline{E}$ is also a limit point of E. Let x be a limit point of $\overline{E}$ and let r be any positive real number.

We want to show that the neighborhood $N_r(x)$ must contain a point in E distinct from x. But we know that $N_r(x)$ contains a point $y \in \overline{E}$ with $y \neq x$. So $y \in E$ or $y \in E'$. If $y \in E$ then we are done, so suppose $y \in E'$ but $y \notin E$.

Then y is a limit point of E, so every neighborhood of y must contain a point in E. In particular, let

$$s = \frac{r - d(x, y)}{2},$$

and choose $z \in N_s(y)$ such that $z \in E$. Then since $N_s(y) \subset N_r(x)$, we have $z \in N_r(x)$ and $z \in E$. And $z \neq x$ since $x \notin N_s(y)$. So x is a limit point of E.

Now we show the converse. Let x be a limit point of E. Then every neighborhood of x contains a point in E distinct from x, but this point must also be in $\overline{E} = E \cup E'$. Therefore x is a limit point of $\overline{E}$.

We have shown that E and $\overline{E}$ have exactly the same set of limit points. That is, $E' = (\overline{E})'$.

Next, to show that E' is closed, let x be a limit point of E'. Then $x \in \overline{E}$. But $\overline{E}$ is closed by Theorem 2.27, so $\overline{E} = (\overline{E})'$. Therefore $x \in (\overline{E})' = E'$. Thus every limit point of E' is in E', so the set E' is closed.

Lastly, it is not the case that E and E' must have the same limit points. For example, take $E = \{1/n \mid n \in Z^+\}$, where Z^+ denotes the positive integers. Then $E' = \{0\}$ but $(E')'$ is the empty set. □

2.7 Exercise 7

Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \ldots$.

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Solution. (a) Let B_n be as stated, and suppose $x \in \overline{B_n}$. Then either $x \in B_n$ or $x \in B_n'$. First, if $x \in B_n$, then $x \in A_i$ for some index i and we have $x \in \overline{A_i}$ so that $x \in \bigcup_i \overline{A_i}$. Now suppose instead that $x \in B_n'$. We want to show that $x \in \overline{A_i}$ for some i. Let $N_r(x)$ be any neighborhood of x, and choose a point $y \neq x$ in this neighborhood such that $y \in B_n$ (this is possible since x is a limit point of B_n). Then $y \in A_i$ for some index i, and such a y can be found for any neighborhood of x, so x is a limit point of A_i. That is, $x \in \overline{A_i}$. This shows that

$$\overline{B_n} \subset \bigcup_{i=1}^n \overline{A_i}. \tag{2.2}$$

Next, suppose $x \in \bigcup_i \overline{A_i}$, so that $x \in \overline{A_i}$ for some index i. Then $x \in A_i$ or $x \in A_i'$. If $x \in A_i$ then $x \in \overline{B_n}$ and we are done. So suppose $x \in A_i'$. Let $N_r(x)$ be any neighborhood of x, and choose $y \neq x$ such that $y \in N_r(x) \cap A_i$. Then $y \in B_n$, which proves that $x \in \overline{B_n}$. This shows that

$$\overline{B_n} \supset \bigcup_{i=1}^n \overline{A_i}. \tag{2.3}$$

Together, (2.2) and (2.3) show that $\overline{B_n} = \bigcup_i \overline{A_i}$. □

(*b*) Suppose $x \in \bigcup_i \overline{A_i}$. Then there is a positive integer i such that $x \in \overline{A_i}$. This implies that $x \in A_i$ or $x \in A_i'$. If $x \in A_i$, then $x \in B$ so certainly $x \in \overline{B}$. On the other hand, if $x \in A_i'$, then for any neighborhood $N_r(x)$, we may find $y \neq x$ in this neighborhood such that $y \in A_i$. Then $y \in B$, which proves that x is a limit of point of B. So in either case, $x \in \overline{B}$, and the inclusion $\overline{B} \supset \bigcup_i \overline{A_i}$ is proved.

Lastly, we show that this inclusion can be proper. For each positive integer i, let $A_i = \{1/i\}$. That is, let each A_i contain only one point, namely the reciprocal of the index. Then each $\overline{A_i}$ also consists of only this one point, so $\bigcup_{i=1}^{\infty} \overline{A_i}$ is the set $\{1/i \mid i \in Z^+\}$. However, $\overline{B}$ contains the point 0, which is not in $\bigcup \overline{A_i}$. □

2.8 Exercise 8

Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.

Solution. We will show that every point of every open set E in R^2 is a limit point of E. Let $\mathbf{x} \in E$. Since E is open, $\mathbf{x}$ is an interior point, and we can find a neighborhood $N_r(\mathbf{x}) \subset E$. Since $E \subset R^2$, there are infinitely many points in $N_r(\mathbf{x})$ distinct from $\mathbf{x}$, and this is still true if we use a smaller positive value for r. Therefore every neighborhood of $\mathbf{x}$ contains a point in E distinct from $\mathbf{x}$. This means that $\mathbf{x}$ is a limit point of E. It follows that every point in E is a limit point of E.

The same is not true for closed sets in R^2. For example, the set

$$E = \{(0,0)\} \cup \{(1/n, 0) \in R^2 \mid n \in Z^+\}$$

contains its only limit point $(0,0)$ and is thus closed. However, $(1,0) \in E$ but $(1,0) \notin E'$. □

2.9 Exercise 9

Let E° denote the set of all interior points of a set E.

(*a*) Prove that E° is always open.

Proof. Let $x \in E^\circ$. Then x is an interior point of E, and we can find a neighborhood $N_r(x) \subset E$. Let y be any point in $N_r(x)$ and let

$$s = \frac{r - d(x,y)}{2}.$$

Then $N_s(y) \subset N_r(x) \subset E$, so y is itself an interior point of E. This shows that $N_r(x) \subset E^\circ$, so x is an interior point of E°. And x was chosen to be arbitrary, so this shows that every point in E° is an interior point, hence E° is open. □

(*b*) Prove that E is open if and only if $E^\circ = E$.

Proof. If $E^\circ = E$ then every point of E is an interior point, and E is open by definition. The converse also follows directly from the definitions: if E is open then $E \subset E^\circ$; moreover, every interior point of E must be in E, so $E^\circ \subset E$ and therefore $E = E^\circ$. □

(*c*) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Proof. Let $x \in G$ be arbitrary. Since G is open, we can find a neighborhood $N_r(x) \subset G$. But $G \subset E$ so $N_r(x) \subset E$. This shows that $x \in E^\circ$ so that $G \subset E^\circ$. □

(*d*) Prove that the complement of E° is the closure of the complement of E.

Proof. First, suppose $x \notin E^\circ$. Then every neighborhood of x must contain a point that is not in E. This means that x is a limit point of E^c so by definition x is in the closure of E^c. This shows that $(E^\circ)^c \subset \overline{E^c}$.

Next, suppose x is in the closure of E^c. Then either x is in E^c or x is a limit point of E^c. In the first case, x cannot be an interior point of E since $x \notin E$. In the second case, every neighborhood of x contains a point in E^c, so x is not an interior point of E. This shows that $x \notin E^\circ$ so that $\overline{E^c} \subset (E^\circ)^c$. This completes the proof that $(E^\circ)^c = \overline{E^c}$. □

(*e*) Do E and $\overline{E}$ always have the same interiors?

Solution. No, E and $\overline{E}$ need not have the same interiors. As a counterexample, consider the nonzero real numbers $E = R - \{0\}$. Clearly 0 is not an interior point of E, yet it is an interior point of $\overline{E} = R$. □

(*f*) Do E and E° always have the same closures?

Solution. No, for example in R^1 if $E = \{0\}$ then $0 \in \overline{E}$, however E° is the empty set and so is its closure. □

2.10 Exercise 10

Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution. If $p \neq q$ then $d(p,q) = 1 > 0$, and $d(p,p) = 0$. It is also clear that $d(p,q) = d(q,p)$. It remains to be shown that

$$d(p,q) \leq d(p,r) + d(r,q) \tag{2.4}$$

for any $r \in X$. If $p = q$ then the result is obvious, so suppose $p \neq q$. Then the right-hand side of the inequality (2.4) is at least 1, and the left-hand side is exactly 1. This shows that d is a metric.

Every subset of X is open, since every point p in a set $E \subset X$ is an interior point (choose $r = 1/2$ to get a neighborhood contained in E).

Every subset of X is also closed, since any such set has no limit points, so it is vacuously true that every limit point of E is in E.

Finally, every finite subset of X is clearly compact. But every infinite subset is not compact, as we will now show. Let E be an infinite subset of X. For each $x \in E$, define $G_x = \{x\}$. Then each G_x is open and $E \subset \bigcup_{x \in E} G_x$ so $\{G_x\}$ is an open cover of E, but it does not have a finite subcover. □

2.11 Exercise 11

For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution. (*a*) For $d_1(x,y) = (x-y)^2$, the first two parts of the definition are satisfied. However, $d_1(1,3) = 4 \not\leq 2 = d(1,2) + d(2,3)$. So d_1 is not a metric.

(*b*) For $d_2(x,y) = \sqrt{|x-y|}$, we clearly have $d(x,y) > 0$ for $x \neq y$, $d(x,x) = 0$, and $d(x,y) = d(y,x)$. Now, by the triangle inequality, we have for any $z \in R^1$,

$$\begin{aligned} d(x,y)^2 &= |x-y| \\ &= |x - z + z - y| \\ &\leq |x-z| + |z-y| \\ &= d(x,z)^2 + d(z,y)^2 \\ &\leq d(x,z)^2 + 2d(x,z)d(z,y) + d(z,y)^2 \\ &= (d(x,z) + d(z,y))^2, \end{aligned}$$

and by taking square roots we have $d(x,y) \leq d(x,z) + d(z,y)$. Therefore d_2 is a metric.

(*c*) For $d_3(x,y) = |x^2 - y^2|$ we have $d_3(-1,1) = 0$, so d_3 is not a metric.

(*d*) For $d_4(x,y) = |x - 2y|$, we have $d(0,1) = 2 \neq 1 = d(1,0)$ so d_4 is not a metric.

(e) For $d_5(x,y) = |x-y|/(1+|x-y|)$, it is clear that $d(x,y) > 0$ for $x \neq y$, $d(x,x) = 0$, and $d(x,y) = d(y,x)$. It remains to be shown that $d(x,y) \leq d(x,z) + d(z,y)$ for all $x \in R^1$. That is, we need to show that

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}. \tag{2.5}$$

Put $a = |x-y|$, $b = |x-z|$, and $c = |z-y|$. Then (2.5) becomes

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Multiplying through by the product of the denominators, we get

$$a(1+b)(1+c) \leq b(1+a)(1+c) + c(1+a)(1+b).$$

Expanding then gives

$$a + ab + ac + abc \leq b + c + ab + ac + 2bc + 2abc$$

which reduces to $a \leq b + c + 2bc + abc$, which follows from the triangle inequality after back-substituting for a, b, and c. So (2.5) holds and d_5 is a metric. □

2.12 Exercise 12

Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine–Borel theorem).

Proof. Let $\{G_\alpha\}$ be any open cover of K. Then there is an index α_1 such that $0 \in G_{\alpha_1}$. Then since G_{α_1} is open, 0 is an interior point so that there is a segment (a,b) containing 0 that lies within G_{α_1}. But there are at most only finitely many values in K which do not belong to this segment ($1/n \geq b$ for only finitely many choices of n). Label these values $r_2, r_3, r_4 \ldots, r_k$. Then $r_i \in G_{\alpha_i}$ for some index α_i ($i = 2, 3, \ldots, k$). Now it is clear that

$$K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$$

so $\{G_{\alpha_i}\}$ is a finite subcover and K is compact. □

2.13 Exercise 13

Construct a compact set of real numbers whose limit points form a countable set.

Solution. For each positive integer k, set

$$\alpha_k = \frac{2^k - 1}{2^k}.$$

That is, $\{\alpha_k\}$ is the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$. Now, for each positive integer k, define the set A_k by

$$A_k = \{\alpha_k\} \cup \left\{ \alpha_k + \frac{1}{2^{k+2}n} \;\middle|\; n = 1, 2, 3, \ldots \right\}.$$

Now, consider the set

$$K = \bigcup_{k=1}^{\infty} A_k.$$

We claim that K is compact and has a countable set of limit points. First, note that K is bounded, since $K \subset (0, 1)$.

Next we show that the set of limit points of K is precisely the set

$$\{\alpha_k \mid k = 1, 2, \ldots\}.$$

It is clear that α_k is a limit point for all positive integers k. Now suppose γ is any other limit point of K. Let (a, b) be a segment containing γ, and we may make this segment small enough so that it is contained entirely within the segment (α_i, α_{i+1}) for some positive integer i. But now this segment (a, b) must contain only finitely many elements from K, since $\alpha_i + 1/(2^{i+2}n) < a$ for sufficiently large n. Label these elements $\beta_1, \ldots, \beta_j$ and set

$$\delta = \min_{1 \le k \le j} \frac{|\gamma - \beta_k|}{2}.$$

Then consider the segment $S = (\gamma - \delta, \gamma + \delta)$. It is clear that γ is the only member of K within S. But γ is a limit point of K, so we can find $\alpha \in S$ such that $\gamma \neq \alpha$ and $\alpha \in K$. This is a contradiction, so $\alpha_1, \alpha_2, \ldots,$ are the only limit points of K.

K contains all of its limit points, so this shows that K is closed and bounded and hence compact (since $K \subset R$), and its limit points are the countable set $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$. □

2.14 Exercise 14

Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution. For each positive integer n, set A_n to be the segment

$$A_n = \left(\frac{1}{n+2},\ 1 - \frac{1}{n+2} \right).$$

Then the sequence $\{A_n\}$ is an open cover of $(0, 1)$ with no finite subcover. □

2.15 Exercise 15

Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."

Theorem. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Corollary. *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \dots)$, then $\bigcap_1^\infty K_n$ is not empty.*

Solution. We will consider counterexamples in R^1.

First we look at closed sets. For each positive integer n, let

$$K_n = \{x \in R \mid x \geq n\}.$$

Then $\{K_n\}$ is a collection of closed sets and every finite subcollection has a nonempty intersection. We also have $K_n \supset K_{n+1}$ for each n. However, for any $x \in R$ we need only choose $n > x$ so that $x \notin K_n$. Therefore $\bigcap_{n=1}^\infty K_n$ is the empty set. So merely being closed is not a sufficient condition for the theorem or its corollary.

Now we consider boundedness. For each positive integer n, set

$$K_n = (0, 1/n).$$

Then $\{K_n\}$ is a collection of bounded sets, every finite subcollection has a nonempty intersection, and $K_n \supset K_{n+1}$ for each n. Suppose $x \in \bigcap_{n=1}^\infty K_n$. Then $0 < x < 1/n$ for each integer n, which is impossible since R is archimedean. Therefore $\bigcap_{n=1}^\infty K_n$ is the empty set. So boundedness is also not sufficient for the theorem or its corollary. □

2.16 Exercise 16

Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

Solution. For the moment, consider the metric space R^1. Taking E as a subset of R, it is clear that E is bounded. Moreover the set of limit points of E is precisely the interval $[2^{1/2}, 3^{1/2}]$. Since E is defined to be the set of rational numbers in this interval, it is clear that E contains all of its rational limit points.

Now consider Q as the metric space (with the same metric, $d(p, q) = |p - q|$). Since the metric is the same, E is still a bounded set. And E contains all of its limit points (since we are only considering rational numbers). This shows that E is closed in Q.

E is closed and bounded in Q, but it is not compact. For example, for each positive integer n, let

$$A_n = \left\{ x \in Q \;\middle|\; x^2 > 2 + \frac{1}{n+2} \text{ and } x^2 < 3 - \frac{1}{n+2} \right\}.$$

Then it is easy to verify that $\{A_n\}$ is an open cover of E which has no finite subcover.

Lastly, we note that E is also open in Q, since every point $p \in E$ is an interior point. Indeed, if $p^2 > 2$ then we can find a $q \in Q$ with $2 < q^2 < p^2$ and similarly if $p^2 < 3$ then we can find an $r \in Q$ with $p^2 < r^2 < 3$. Then $p \in (q, r) \cap Q \subset E$ and we can find a neighborhood of p contained entirely within this segment. □

2.17 Exercise 17

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Solution. E is not countable. For, if it is countable, let $f\colon Z^+ \to E$ be a 1-1 correspondence. Then consider the real number $a \in [0,1]$ whose nth digit is a 4 if the corresponding digit in $f(n)$ is a 7, or a 7 if the corresponding digit in $f(n)$ is a 4. $a \in E$ but there is no positive integer n such that $f(n) = a$, so this gives a contradiction.

E is not dense in $[0,1]$ since $E \subset [0.4, 0.8]$, so 0.1 (for example) is neither a member of E nor a limit point of E.

E is compact, since it is a closed and bounded subset of the compact space $[0,1]$. To prove that E is closed, we will show that its complement is open. So let $x \in E^c$. That is, $x \in [0,1]$ is such that the decimal expansion of x contains a digit other than 4 and 7. Suppose the kth digit of x is not 4 or 7 and let $\delta = 10^{-k-2}$. Then all of the numbers in the segment $(x-\delta, x+\delta)$ do not differ from x in their first k digits, so that their kth digit is not 4 or 7. So the segment $(x-\delta, x+\delta) \subset E^c$, which shows that x is an interior point of E^c. Therefore E^c is open, so E is closed.

Lastly, we show that E is perfect. We already know E is closed, so it remains to be shown that every point in E is a limit point. Let $x \in E$, and for $r > 0$ let $(x-r, x+r)$ be any neighborhood of x. Choose k large enough so that $10^{-k} < r$. Let y be the number formed by "flipping" the $(k+2)$th digit of x (i.e., if the $(k+2)$th digit is a 4 make it a 7 and vice versa). Then $y \neq x$, $y \in E$, and $y \in (x-r, x+r)$, so this shows that x is a limit point of E. Therefore every point in E is a limit point and E is perfect. □

2.19 Exercise 19

(*a*) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof. By Theorem 2.27, we have $A = \overline{A}$ and $B = \overline{B}$. So, since $A \cap B$ is empty, we have that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. □

(*b*) Prove the same for disjoint open sets.

Proof. Let A and B be disjoint open sets and suppose $A \cap \overline{B}$ is nonempty. Let $x \in A \cap \overline{B}$, so that x is a limit point of B. Since $x \in A$ and A is open, there is a neighborhood of x contained entirely within A. But this neighborhood must contain points of B, since x is a limit point of B. This shows that A and B are not disjoint, which gives a contradiction. □

(*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

Proof. A and B are both disjoint open sets, so they must be separated by the previous part of the exercise. □

(d) Prove that every connected metric space with at least two points is uncountable.

Proof. Let E be a connected metric space with at least two points, p and q. Let $s = d(p, q) > 0$. We will form a one-to-one correspondence $f\colon (0, s) \to A$ where $A \subset E$. This will show that A, and hence E, is uncountable.

For each δ in the segment $(0, s)$, choose any point $r \in E$ such that $d(p, r) = \delta$ and then set $f(\delta) = r$. We know that the point r must exist, for if not then we could divide E into two subsets, those points x with $d(p, x) < \delta$ and those y with $d(p, y) > \delta$ so that, by the previous part of this exercise, E is not connected.

Therefore the function f exists. And it gives a one-to-one correspondence since $d(p, r_1) \neq d(p, r_2)$ implies $r_1 \neq r_2$. This completes the proof that E is uncountable. □

2.20 Exercise 20

Are closures and interiors of connected sets always connected?

Solution. Closures of connected sets are connected. For, if not, let E be a connected set whose closure $\overline{E}$ is not connected. Then $\overline{E} = A \cup B$ where A and B are nonempty separated sets. Let $A^* = A \cap E$ and $B^* = B \cap E$.

Since A is nonempty, we may choose $x \in A$. Then $x \in \overline{E}$ so either $x \in E$ or x is a limit point of E. If the latter, then any neighborhood of x must contain a point y in E. Moreover it must be possible to choose y so that $y \in A$. If not, then x is a limit point of B, so that $A \cap \overline{B}$ is nonempty, which is a contradiction. So A contains a point in E and therefore A^* is nonempty. By the same argument, B^* is nonempty. And any $x \in E$ must belong to A or B and hence to A^* or B^*, so that E is the union of the two nonempty separated sets A^* and B^*, which contradicts the fact that E is connected. This shows that $\overline{E}$ must be connected.

However, the interior of a connected set need not be connected. Consider the space R^2 and take

$$E = \{(x, y) \in R^2 \mid \sqrt{(x+1)^2 + y^2} \leq 1\} \cup \{(x, y) \in R^2 \mid \sqrt{(x-1)^2 + y^2} \leq 1\}.$$

That is, E is the union of the two closed disks of radius 1 centered at $(-1, 0)$ and $(1, 0)$, respectively. The interior of this set is the union of the corresponding open disks, and these open disks are separated since the point of tangency $(0, 0)$ is not an interior point of E. Therefore the interior of E is not connected. □

2.21 Exercise 21

Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of R^1.

Proof. First A_0 and B_0 must be disjoint. If not, let $x \in A_0 \cap B_0$. Then $\mathbf{p}(x) \in A$ and $\mathbf{p}(x) \in B$ so that $A \cap B$ is nonempty, which is a contradiction.

Next, we must show that $A_0 \cap \overline{B_0}$ is empty. Suppose $y \in A_0$ and $y \in \overline{B_0}$. Then y is a limit point of B_0, so that any segment in R^1 containing y must contain points of B_0. Now for some $r > 0$, let $N_r(\mathbf{p}(y))$ be any neighborhood of $\mathbf{p}(y)$. Set

$$\delta = \frac{r}{d(\mathbf{a}, \mathbf{b})}.$$

Then the segment $(y - \delta, y + \delta)$ contains a point z in B_0. Then $\mathbf{p}(z) \in B$. But

$$\begin{aligned} d(\mathbf{p}(y), \mathbf{p}(z)) &= |\mathbf{p}(y) - \mathbf{p}(z)| \\ &= |(1-y)\mathbf{a} + y\mathbf{b} - (1-z)\mathbf{a} - z\mathbf{b}| \\ &= |(z-y)\mathbf{a} + (y-z)\mathbf{b}| \\ &= |y-z||\mathbf{b} - \mathbf{a}| \\ &\leq \delta d(\mathbf{a}, \mathbf{b}) = r. \end{aligned}$$

Thus $\mathbf{p}(z) \in N_r(\mathbf{p}(y))$. And this neighborhood was arbitrary, so that $\mathbf{p}(y)$ is a limit point of B. Therefore $A \cap \overline{B}$ is nonempty, which is a contradiction. This shows that A_0 and $\overline{B_0}$ are disjoint. By the same argument, B_0 and $\overline{A_0}$ are disjoint. This shows that A_0 and B_0 are separated sets. □

(*b*) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

Proof. $0 \in A_0$ and $1 \in B_0$. And we have shown that A_0 and B_0 are separated. So by Theorem 2.47 there exists $t_0 \in (0, 1)$ such that $t_0 \notin A_0 \cup B_0$. Then by definition $\mathbf{p}(t_0) \notin A \cup B$. □

(*c*) Prove that every convex subset of R^k is connected.

Proof. Suppose not, and let E be a convex subset of R^k that is not connected. E can be written as the union of two nonempty separated sets A and B. Choose $\mathbf{a} \in A$ and $\mathbf{b} \in B$. Define $\mathbf{p}\colon R \to R^k$ as above, with $A_0 = \mathbf{p}^{-1}(A)$ and $B_0 = \mathbf{p}^{-1}(B)$. By the previous results of this exercise, A_0 and B_0 must be separated and there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$. But this contradicts the fact that E is convex. Therefore, E must be connected, so that every convex subset of R^k is connected. □

2.22 Exercise 22

A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable.

Proof. The set Q^k consisting of points in R^k having only rational coordinates is dense in R^k (since Q is dense in R). Q^k is also countable, being the Cartesian product of countable sets. Therefore R^k is separable. □

2.23 Exercise 23

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base.

Proof. Let X be a separable metric space. Then X contains a countable dense subset E. For each $p \in E$ and for each rational number $s > 0$, let $V_{s,p}$ be the neighborhood $N_s(p)$ of radius s centered at p. Then $\{V_{s,p}\}$ is a countable collection of open subsets of X.

Let $x \in X$ be arbitrary and let G be any open subset of X containing x. Since G is open, there is a neighborhood $N_r(x)$ of x with radius $r > 0$ contained entirely within G. Take the smaller neighborhood $N_{r/2}(x)$, and choose a point x^* within this neighborhood such that $x^* \in E$ (this is possible since E is dense in X). Now, let

$$\delta = d(x, x^*).$$

Choose any rational number r^* in the segment $(\delta, 2\delta)$. Then the neighborhood V_{r^*,x^*} contains x. The neighborhood is also contained in G. So $x \in V_{r^*,x^*} \subset G$ for any open subset G of X containing the point x. Therefore the collection $\{V_{s,p}\}$ of neighborhoods with rational radius and center in E is a countable base for X. □

2.24 Exercise 24

Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Proof. Fix $\delta > 0$ and pick $x_1 \in X$. Now, having chosen $x_1, \ldots, x_j$, choose, if possible, $x_{j+1} \in X$ such that

$$d(x_i, x_{j+1}) \geq \delta, \quad i = 1, 2, \ldots, j.$$

Continue choosing values in this way until it is no longer possible. We know that this process must terminate after a finite number of steps because otherwise $x_1, x_2, x_3, \ldots,$ would be an infinite subset of X which has no limit point (since each point is isolated), which is a contradiction. Therefore X can be covered by finitely many neighborhoods of radius δ.

Now for each positive integer n, repeat the above procedure using $\delta = 1/n$. For each n there are finitely many neighborhoods, so the centers of these neighborhoods, over all n, form a countable subset E of X. And this subset is dense in X: choose any $x \in X$. If $x \notin E$ then any neighborhood of x of radius r must be covered by smaller neighborhoods of radius $\delta < r$ with centers in E, so that x is a limit point for E.

E is a countable dense subset of X, so X is separable. □

2.25 Exercise 25

Prove that every compact metric space K has a countable base, and that K is therefore separable.

Proof. Let K be a compact metric space. For each positive integer n, let $\delta = 1/n$ and consider the collection of all neighborhoods of radius δ in K. Since K is compact, this open cover must have a finite subcover, so label the centers of the neighborhoods in the finite subcover as $x_{n,1}, x_{n,2}, \ldots, x_{n,k}$. Let

$$V_{n,i} = \left\{ y \in K \;\middle|\; d(x_{n,i}, y) < \frac{1}{n} \right\}.$$

Then the collection $\{V_{n,i}\}$ is a countable base for K. The set $\{x_{n,i} \mid n, i \in Z^+\}$ is a countable dense subset of K, so K is separable. □

2.26 Exercise 26

Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.

Proof. By Exercise 2.24, X is separable, so by Exercise 2.23 X has a countable base. Let $\{G_\alpha\}$ be an open cover of X. Since X has a countable base we may find a countable subcover $\{G_n\}$ for $n = 1, 2, 3, \ldots$.

For each n, let F_n denote the complement of $G_1 \cup G_2 \cup \cdots \cup G_n$. Suppose that no finite subcollection of $\{G_n\}$ covers X. Then each F_n is nonempty, but $\bigcap_{n=1}^{\infty} F_n$ is empty. Let E be a set containing one point from each F_n. It is clear that F_n contains no points from G_m for any $m \geq n$, so any particular G_n must contain only finitely many points belonging to E.

But E is an infinite subset of X, so it has a limit point x. Then $x \in G_n$ for some n, and since G_n is open there is a neighborhood of x contained within G_n. Because x is a limit point, this neighborhood must contain points of E. Since such points must exist no matter how small the radius of the neighborhood is made, it follows that G_n contains infinitely points from E. This contradicts our earlier finding that $G_n \cap E$ is finite. Therefore the open cover $\{G_n\}$ has a finite subcover and X is compact. □

2.27 Exercise 27

Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.

Proof. By Exercise 2.22 we know that R^k is separable, so by Exercise 2.23 we know that it has a countable base $\{V_n\}$.

Let W be the union of those V_n for which $E \cap V_n$ is at most countable. Then W is open since it is a union of open sets. Suppose $x \in W$. Then $x \in V_i$ for some i such that $V_i \cap E$ is at most countable. Then any neighborhood of

x contained within V_i has at most countably many points of E, so x is not a condensation point of E, i.e., $x \notin P$. This shows that $W \subset P^c$.

Conversely, suppose $x \notin P$. Then there is a neighborhood N of x containing at most countably many points of E. So there is a j such that $x \in V_j \subset N$, where $V_j \cap E$ is at most countable. Hence $x \in W$, so that $P^c \subset W$. Therefore $W = P^c$.

And $W \cap E$ is at most countable, since W is a union of countably many sets $\{V_i\}$ and each V_i contains at most countably many points of E.

It remains to be shown that P is perfect. But, since W is open, its complement P is closed. So we need only show that every point in P is a limit point of P.

To that end, let $x \in P$ be arbitrary, let N be any neighborhood of x, and suppose for the purpose of finding a contradiction that no point in N distinct from x is a condensation point of E. Then every point in $N - \{x\}$ is in W. Therefore $N - \{x\}$ contains at most countably many points of E. But this means that N itself contains at most countably many points of E, which contradicts the fact that x is a condensation point. Therefore x is a limit point of P and P is perfect. □

2.28 Exercise 28

Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary*: Every countable closed set in R^k has isolated points.)

Proof. Although the previous exercise concerned R^k, the proof actually works for any separable metric space. So if X is separable and if $E \subset X$, then the set P of all condensation points of E is a perfect set. But E is closed, so it contains all of its limit points, and hence contains its condensation points, so that $P \subset E$. And we know that $E \cap P^c$ is at most countable. Therefore E is the union of a perfect set (which may be empty) and a set which is at most countable. □

2.29 Exercise 29

Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments.

Proof. By Exercise 2.22 R^1 is separable, so it has a countable dense subset Q.

Let E be any open set in R^1. If E is empty, then the result holds vacuously, so suppose E is nonempty, and let $x \in E$. x must be an interior point, so there is a segment contained entirely within E which contains x. In fact there are many such segments, so call the collection of all such segments $\{I_{x,\alpha}\}$ where each α is an index in some set A of indices. Then we may define the *maximal segment* I_x as follows:

$$I_x = \bigcup_{\alpha \in A} I_{x,\alpha}.$$

Now take the collection $I = \{I_x\}$ of all maximal segments in E. We will show that this is an at most countable collection of disjoint segments whose union is E.

First, if the two maximal segments I_x and I_y have any points in common, then there is a segment containing all such points which is contained within E, so by definition $I_x = I_y$. Therefore, if I_x and I_y are distinct, then they must be disjoint.

Next, let I_x be any maximal segment and take a point $p \in I_x$. Since Q is dense in R, p is either in Q or is a limit point of Q. If it is a limit point, then by definition any segment containing p must contain points from Q. Either way, the maximal segment I_x contains points in Q. Since Q is countable, and since each member of Q belongs to at most one maximal segment I_x (since the segments are disjoint), this shows that $\{I_x\}$ is a countable collection of segments. Hence we may use the positive integers as subscripts so that the maximal segments in E can be labeled $I_1, I_2, I_3, \ldots$.

Finally, it is clear that $\bigcup_{n=1}^{\infty} I_n = E$ since every point in E is contained within some segment (because E is open), and each maximal segment is contained within E by construction. Therefore E is the union of an at most countable collection $\{I_n\}$ of disjoint segments. □

2.30 Exercise 30

Imitate the proof of Theorem 2.43 to obtain the following result:

> If $R^k = \bigcup_1^{\infty} F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.
>
> *Equivalent statement:* If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots$, then $\bigcap_1^{\infty} G_n$ is not empty (in fact, it is dense in R^k).

Proof. We first will establish the equivalence of the two statements. Suppose the first statement is true and let G_n be a dense open subset of R^k for each $n = 1, 2, 3, \ldots$. If $F_n = G_n^c$ for each n, then each F_n is a closed subset of R^k. Fix some n and suppose F_n has a nonempty interior, so that $\mathbf{x}$ is an interior point. Then there is a neighborhood of $\mathbf{x}$ containing no points in G_n. But then $\mathbf{x}$ is neither in G_n nor a limit point of G_n, which contradicts the fact that G_n is dense in R^k. Hence F_n has no interior points. And this is true for each n, so that by hypothesis, $\bigcup F_n$ cannot be equal to R^k. This shows that $\bigcap G_n$ is nonempty.

Now suppose the second statement is true, and let F_n be a closed subset of R^k whose union is R^k. Take $G_n = F_n^c$ for each n, so that each G_n is an open subset of R^k. Since $R^k = \bigcup F_n$, we must have that $\bigcap G_n$ is empty, so that G_n is not dense in R^k for at least one n. But then there is a point $\mathbf{x}$ in F_n which is not a limit point of G_n, so that $\mathbf{x}$ is an interior point of F_n. Hence F_n has a nonempty interior for at least one n. So the two statements are equivalent.

We will now prove the second statement. Let G_n be a dense open subset of R^k for each positive integer n. Take any point $\mathbf{x}_1 \in G_1$. Since G_1 is open, there is a neighborhood of $\mathbf{x}_1$ contained within G_1, and within this neighborhood we can find a closed ball B_1 centered at $\mathbf{x}_1$ so that $B_1 \subset G_1$.

Having constructed the closed ball $B_n \subset G_n$, centered on the point $\mathbf{x}_n$, take any point $\mathbf{p}$ in the interior of B_n distinct from $\mathbf{x}_n$. Since G_{n+1} is dense in R^k, $\mathbf{p}$

must be in G_{n+1} or is a limit point of G_{n+1}. If the former, set $\mathbf{x}_{n+1} = \mathbf{p}$. If the latter, any neighborhood of $\mathbf{p}$ must contain points from G_{n+1}, so we may take $\mathbf{x}_{n+1}$ to be any such point that is contained within the interior of B_n. Now $\mathbf{x}_{n+1}$ is an interior point of G_{n+1} and B_n, so there is a closed ball B_{n+1} contained within $B_n \cap G_{n+1}$.

We have constructed a sequence $\{B_n\}$ of nonempty sets that are each closed and bounded (and hence compact), with $B_1 \supset B_2 \supset B_3 \supset \cdots$. Therefore by the Corollary to Theorem 2.36, $\bigcap_1^\infty B_n$ is nonempty. But each $B_n \subset G_n$, so that $\bigcap G_n$ is nonempty as well. $\square$

Chapter 3

Numerical Sequences and Series

3.1 Exercise 1

Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Proof. Suppose $\{s_n\}$ converges to s for some complex sequence $\{s_n\}$ and $s \in C$. Let $\varepsilon > 0$ be arbitrary. Then we may find N such that $|s_n - s| < \varepsilon$ for all $n \geq N$. Then, by Exercise 1.13 we have

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon \quad \text{for each } n \geq N.$$

Hence $\{|s_n|\}$ converges to $|s|$.

Note that the converse is *not* necessarily true. For example the real sequence $\{a_n\}$ given by $a_n = (-1)^n$ does not converge even though $\{|a_n|\}$ converges to 1. □

3.2 Exercise 2

Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Solution. We have

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt{n^2+n} - n) &= \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} \\
&= \frac{1}{2}.
\end{aligned}$$

□

3.3 Exercise 3

If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

Proof. We will show by induction on n that $\{s_n\}$ is a strictly increasing sequence that is bounded above by 2. Certainly $\sqrt{2} < \sqrt{2 + \sqrt{2}} < 2$, so the base case is satisfied. Suppose $s_n < s_{n+1} < 2$ for a positive integer n. Then

$$s_{n+2} = \sqrt{2 + \sqrt{s_{n+1}}} > \sqrt{2 + \sqrt{s_n}} = s_{n+1}$$

and

$$s_{n+2} = \sqrt{2 + \sqrt{s_{n+1}}} < \sqrt{2 + \sqrt{2}} < \sqrt{4} = 2.$$

Therefore $s_{n+1} < s_{n+2} < 2$ and it follows that $\{s_n\}$ is monotonic and bounded, and hence must converge. □

3.4 Exercise 4

Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution. $\{s_n\}$ is the sequence

$$0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \dots.$$

The odd terms of $\{s_n\}$ form the sequence

$$0, \frac{1}{4}, \frac{3}{8}, \frac{7}{16}, \dots, \frac{2^{n-1}-1}{2^n}, \dots$$

while the even terms form the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots, \frac{2^n - 1}{2^n}, \dots.$$

So

$$\begin{aligned} \liminf_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{2^{n-1}-1}{2^n} \\ &= \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2^n}\right) \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{2^n - 1}{2^n} \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) \\ &= 1. \end{aligned}$$

□

3.5 Exercise 5

For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. For each positive integer n, put $c_n = a_n + b_n$. Let

$$\alpha = \limsup_{n\to\infty} a_n, \quad \beta = \limsup_{n\to\infty} b_n, \quad \text{and} \quad \gamma = \limsup_{n\to\infty} c_n.$$

If $\alpha = \infty$ and $\beta \ne -\infty$ then the result is clear, and the case where $\alpha = -\infty$ and $\beta \ne \infty$ is similar.

So suppose α and β are both finite. Let $\{c_{n_i}\}$ be a subsequence of $\{c_n\}$ that converges to γ. Now let $\{a_{n_{i_j}}\}$ be a subsequence of $\{a_{n_i}\}$ such that

$$\lim_{j\to\infty} a_{n_{i_j}} = \limsup_{i\to\infty} a_{n_i}.$$

Now since $\{c_{n_{i_j}}\}$ is a subsequence of $\{c_{n_i}\}$, it converges to the same limit γ. Then

$$\lim_{j\to\infty} b_{n_{i_j}} = \lim_{j\to\infty}(c_{n_{i_j}} - a_{n_{i_j}}) = \lim_{j\to\infty} c_{n_{i_j}} - \lim_{j\to\infty} a_{n_{i_j}} = \gamma - \limsup_{i\to\infty} a_{n_i}.$$

Rearranging, we get

$$\gamma = \limsup_{i\to\infty} a_{n_i} + \lim_{j\to\infty} b_{n_{i_j}}.$$

But

$$\limsup_{i\to\infty} a_{n_i} \le \alpha \quad \text{and} \quad \lim_{j\to\infty} b_{n_{i_j}} \le \beta,$$

so

$$\gamma \le \alpha + \beta$$

and the proof is complete. □

3.6 Exercise 6

Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(*a*) $a_n = \sqrt{n+1} - \sqrt{n}$

Solution. Let s_n denote the nth partial sum of $\sum a_n$. A simple induction argument will show that $s_n = \sqrt{n+1} - \sqrt{1}$. Since $s_n \to \infty$ as $n \to \infty$, the series $\sum a_n$ diverges. □

(*b*) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

Solution. We have

$$a_n = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n\sqrt{n+1}+n\sqrt{n}}.$$

so

$$a_n \leq \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}}.$$

So by comparison (Theorem 3.25) with the convergent series $\sum 1/n^{3/2}$, we see that $\sum a_n$ converges. □

(*c*) $a_n = (\sqrt[n]{n} - 1)^n$

Solution. By Theorem 3.20 (c),

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = 1 - 1 = 0.$$

Therefore, by the root test (Theorem 3.33), the series $\sum a_n$ converges. □

(*d*) $a_n = \dfrac{1}{1+z^n}$ for complex values of z

Solution. First note that, by Exercise 1.13, we have

$$|z^n + 1| = |z^n - (-1)| \geq ||z^n| - |-1|| = ||z|^n - 1|.$$

Then

$$\left|\frac{1}{1+z^n}\right| \leq \frac{1}{||z|^n - 1|}. \tag{3.1}$$

Now suppose $|z| > 1$. Then there is an integer N such that $|z|^n > 2$ for all $n \geq N$. That is,

$$\frac{1}{|z|^n - 1} \leq \frac{2}{|z|^n} \quad \text{for } n \geq N.$$

Using this fact, (3.1) becomes

$$\left|\frac{1}{1+z^n}\right| \leq \frac{2}{|z|^n} \quad \text{for } n \geq N.$$

So by the comparison test with the convergent geometric series $\sum 2/|z|^n$ we have that $\sum a_n$ also converges.

In the case where $|z| \leq 1$, it is easy to see that $a_n \not\to 0$ as $n \to \infty$, so $\sum a_n$ diverges. □

3.7 Exercise 7

Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof. Since

$$\left(\sqrt{a_n} + \frac{1}{n}\right)^2 \geq 0,$$

we may expand and rearrange to get

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right). \tag{3.2}$$

Since $\sum a_n$ and $\sum 1/n^2$ both converge, we know by Theorem 3.47 that their sum,

$$\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2}\right),$$

also converges. By the comparison test, (3.2) implies that $\sum \sqrt{a_n}/n$ must converge. □

3.8 Exercise 8

If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Let A_n denote the nth partial sum of $\sum a_n$. That is,

$$A_n = \sum_{k=1}^{n} a_k.$$

Suppose $\{A_n\}$ converges to A. We know that $\{b_n\}$ must converge, so set $b = \lim_{n\to\infty} b_n$. Then

$$\lim_{n\to\infty} A_n b_n = Ab.$$

Now, since $\{A_n\}$ converges, it must be bounded, so we can find M_0 such that $|A_n| < M_0$ for all n. We can also find M_1 such that $|b_n| < M_1$ for all n. Take $M = \max(M_0, M_1)$. Then for any $\varepsilon > 0$, we may find N such that

$$|A_n b_n - Ab| < \frac{\varepsilon}{3} \quad \text{and} \quad |b_m - b_n| < \frac{\varepsilon}{3M} \quad \text{for all } m, n \geq N.$$

Since $\{b_n\}$ is monotonic, we have for all $q > p > N$ that

$$\begin{aligned}\left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| &\leq M\left|\sum_{n=p}^{q-1}(b_n - b_{n+1})\right| \\ &= M|b_p - b_q| < \frac{\varepsilon}{3},\end{aligned}$$

and

$$\begin{aligned}|A_q b_q - A_{p-1} b_p| &= |A_q b_q - Ab + Ab - A_{p-1} b_p| \\ &\leq |A_q b_q - Ab| + |A_{p-1} b_p - Ab| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.\end{aligned}$$

Finally, using the partial summation formula from Theorem 3.41, we have

$$\begin{aligned}
\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\
&\le \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| + |A_q b_q - A_{p-1} b_p| \\
&< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

By the Cauchy criterion, $\sum a_n b_n$ converges. □

3.9 Exercise 9

Find the radius of convergence of each of the following power series:

(*a*) $\sum n^3 z^n$

Solution. Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(n+1)^3 z^{n+1}}{n^3 z^n}\right| = \lim_{n\to\infty}\frac{n+1}{n}|z| = |z|.$$

So the series converges when $|z| < 1$. Therefore the radius of convergence is $R = 1$. □

(*b*) $\sum \frac{2^n}{n!} z^n$

Solution. Applying the ratio test again, we get,

$$\lim_{n\to\infty}\left|\frac{2^{n+1} z^{n+1} n!}{2^n z^n (n+1)!}\right| = \lim_{n\to\infty}\frac{2}{n+1}|z| = 0,$$

so $R = \infty$. □

(*c*) $\sum \frac{2^n}{n^2} z^n$

Solution. We have

$$\lim_{n\to\infty}\left|\frac{2^{n+1} z^{n+1} n^2}{2^n z^n (n+1)^2}\right| = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2 |z| = 2|z|$$

so $R = 1/2$. □

(*d*) $\sum \frac{n^3}{3^n} z^n$

Solution. Again,

$$\lim_{n\to\infty}\left|\frac{(n+1)^3 z^{n+1} 3^n}{n^3 z^n 3^{n+1}}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 |z| = \frac{1}{3}|z|,$$

so $R = 3$. □

3.10 Exercise 10

Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Proof. For contradiction, suppose that $|z| > 1$ while the sum

$$\sum_{n=0}^{\infty} a_n z^n$$

converges. Then we know that $a_n z^n \to 0$. In particular if $\varepsilon = 1$, we may find an integer N such that

$$|a_n z^n| < 1 \quad \text{for all } n \geq N.$$

But since a_n is an integer and is nonzero for infinitely many n, we may also find $n_0 \geq N$ such that $a_{n_0} \geq 1$. Then

$$|a_{n_0} z^{n_0}| = |a_{n_0}||z^{n_0}| \geq 1.$$

This is a contradiction, so $|z|$ cannot be greater than 1. Therefore the radius of convergence is at most 1. □

3.11 Exercise 11

Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(*a*) Prove that $\sum \frac{a_n}{1 + a_n}$ diverges.

Proof. Suppose the sum converges. Then $a_n/(1 + a_n) \to 0$, which implies that $a_n \to 0$. So we can find a positive integer N such that $a_n < 1$ for all $n \geq N$. Then for $n \geq N$ we have

$$\frac{a_n}{1 + a_n} \geq \frac{a_n}{1 + 1} = \frac{1}{2} a_n.$$

Then the series $\sum a_n/2$ and hence $\sum a_n$ must converge by the comparison test, a contradiction. Therefore $\sum a_n/(1 + a_n)$ must diverge. □

(*b*) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Proof. Since $a_n > 0$, the sequence $\{s_n\}$ is monotonically increasing. Then for any positive integers N and k, we have

$$\begin{aligned} \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}. \end{aligned}$$

This establishes the desired inequality.

Now, since s_n is monotonic, it cannot be bounded (for otherwise $\sum a_n$ would converge). So for any fixed $N > 0$, we may find a positive integer k so that

$$s_{N+k} > 2s_N.$$

Then

$$\sum_{n=1}^{k} \frac{a_{N+n}}{s_{N+n}} \geq 1 - \frac{s_N}{s_{N+k}} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore $\sum a_n/s_n$ diverges by the Cauchy criterion. □

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof. For any n,

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}. \tag{3.3}$$

Now notice that

$$\sum_{n=2}^{k} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{1}{s_1} - \frac{1}{s_k}.$$

Since the right-hand side of this equation tends to $1/s_1$ as $k \to \infty$, we see that the sum on the left converges. By the comparison test with (3.3), it follows that $\sum a_n/s_n^2$ converges. □

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

Solution. The series on the right must converge by comparison with the convergent series $\sum 1/n^2$.

However, the series on the left may converge or diverge depending on the nature of $\{a_n\}$. For example, if $a_n = 1$ for all n, then

$$\frac{a_n}{1 + na_n} = \frac{1}{1 + n},$$

which is just the divergent harmonic series without the first term.

On the other hand, if we set $a_n = 1$ when n is a perfect square and $a_n = 1/n^2$ otherwise, then $\{a_n\}$ does diverge but $\sum a_n/(1 + na_n)$ converges, as we will now show. Let P be the set of perfect squares. Then

$$\begin{aligned}\sum_{n=1}^{m^2}\frac{a_n}{1+na_n} &= \sum_{\substack{1\le n\le m^2\\ n\in P}}\frac{a_n}{1+na_n}+\sum_{\substack{1\le n\le m^2\\ n\notin P}}\frac{a_n}{1+na_n}\\ &= \sum_{n=1}^{m}\frac{1}{1+n^2}+\sum_{\substack{1\le n\le m^2\\ n\notin P}}\frac{1}{n^2+n}\\ &\le \sum_{n=1}^{m}\frac{1}{1+n^2}+\sum_{n=1}^{m^2}\frac{1}{n^2+n}.\end{aligned}$$

If we let $m\to\infty$, then the right-hand side converges, and it follows that $\sum a_n/(1+na_n)$ also converges. □

3.12 Exercise 12

Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m}+\cdots+\frac{a_n}{r_n} > 1-\frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof. Note that the sequence $\{r_n\}$ is strictly decreasing and bounded below by 0. So if $m < n$, we have

$$\frac{a_m}{r_m}+\cdots+\frac{a_n}{r_n} > \frac{a_m+\cdots+a_n}{r_m} = \frac{r_m-r_{n+1}}{r_m} > 1-\frac{r_n}{r_m}.$$

Fix an $m > 0$. Since $r_n\to 0$ we can always find $n > m$ so that $r_n < r_m/2$. Then $1-r_n/r_m > 1/2$. So no matter how large N is, it is possible to find $n > m \ge N$ such that

$$\sum_{k=m}^{n}\frac{a_k}{r_k} > \frac{1}{2},$$

and therefore the series diverges by the Cauchy criterion. □

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n}-\sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Proof. For any index n, we have

$$\begin{aligned}\frac{a_n}{\sqrt{r_n}} &= \frac{r_n - r_{n+1}}{\sqrt{r_n}} \\ &= \frac{(\sqrt{r_n} + \sqrt{r_{n+1}})(\sqrt{r_n} - \sqrt{r_{n+1}})}{\sqrt{r_n}} \\ &= \left(1 + \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right)(\sqrt{r_n} - \sqrt{r_{n+1}}) \\ &< 2(\sqrt{r_n} - \sqrt{r_{n+1}}).\end{aligned}$$

Now observe that, since $r_n \to 0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(\sqrt{r_n} - \sqrt{r_{n+1}}) &= \lim_{m\to\infty}\sum_{n=1}^{m}(\sqrt{r_n} - \sqrt{r_{n+1}}) \\ &= \lim_{m\to\infty}(\sqrt{r_1} - \sqrt{r_{m+1}}) \\ &= \sqrt{r_1}.\end{aligned}$$

This series converges, so by the comparison test, $\sum a_n/\sqrt{r_n}$ also converges. □

3.13 Exercise 13

Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Proof. We will imitate the proof of Theorem 3.50. Suppose $\sum a_n$ and $\sum b_n$ both converge absolutely, define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for each positive integer n, and set

$$\sum_{n=0}^{\infty}|a_n| = A \quad \text{and} \quad \sum_{n=0}^{\infty}|b_n| = B.$$

Further, put

$$A_n = \sum_{k=0}^{n}|a_k|, \quad B_n = \sum_{k=0}^{n}|b_k|, \quad C_n = \sum_{k=0}^{n}|c_k|, \quad \text{and} \quad \beta_n = B_n - B.$$

Then

$$\begin{aligned}C_n &= |a_0b_0| + |a_0b_1 + a_1b_0| + \cdots + |a_0b_n + a_1b_{n-1} + \cdots + a_nb_0| \\ &\le |a_0||b_0| + (|a_0||b_1| + |a_1||b_0|) + \cdots + (|a_0||b_n| + \cdots + |a_n||b_0|) \\ &= |a_0|B_n + |a_1|B_{n-1} + \cdots + |a_n|B_0 \\ &= |a_0|(B + \beta_n) + |a_1|(B + \beta_{n-1}) + \cdots + |a_n|(B + \beta_0) \\ &= A_nB + |a_0|\beta_n + |a_1|\beta_{n-1} + \cdots + |a_n|\beta_0.\end{aligned}$$

Now set

$$\gamma_n = |a_0|\beta_n + |a_1|\beta_{n-1} + \cdots + |a_n|\beta_0.$$

Let $\varepsilon > 0$ be given and note that $\beta_n \to 0$. So we can choose N such that $|\beta_n| \leq \varepsilon$ for $n \geq N$. Then

$$\begin{aligned}|\gamma_n| &\leq |\beta_0|a_n| + \cdots + \beta_N|a_{n-N}|| + |\beta_{N+1}|a_{n-N-1}| + \cdots + \beta_n|a_0|| \\ &\leq |\beta_0|a_n| + \cdots + \beta_N|a_{n-N}|| + \varepsilon A.\end{aligned}$$

Keeping N fixed and letting $n \to \infty$ gives

$$\limsup_{n\to\infty} |\gamma_n| \leq \varepsilon A,$$

since $|a_k| \to 0$ as $k \to \infty$. Since ε was arbitrary, this shows that $\lim_{n\to\infty} \gamma_n = 0$.

Returning to C_n, we have

$$C_n \leq A_n B + \gamma_n.$$

Since $A_n B \to AB$ and $\gamma_n \to 0$, we see that the sequence $\{C_n\}$ is bounded above. But $\{C_n\}$ is a sequence of partial sums for a series in which every term is nonnegative, so the sequence $\{C_n\}$ is also monotonically increasing. Therefore $\sum |c_n|$ converges, so $\sum c_n$ converges absolutely. □

3.15 Exercise 15

Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting.

Theorem. *$\sum \mathbf{a}_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that*

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leq \varepsilon$$

if $m \geq n \geq N$.

Proof. Consider the sequence $\{\mathbf{a}_n\}$ in R^t given by

$$\mathbf{a}_n = (a_{1,n}, a_{2,n}, \ldots, a_{t,n}).$$

Combining Theorem 3.4 with the original Theorem 3.22, we have that $\sum \mathbf{a}_n$ converges if and only if for each $\varepsilon_i > 0$ $(i = 1, 2, \ldots, t)$ there is N_i such that

$$\left|\sum_{k=n}^{m} a_{i,k}\right| \leq \varepsilon_i \quad \text{for all } m \geq n \geq N_i. \tag{3.4}$$

Suppose (3.4) holds and let $\varepsilon > 0$. For each i, set $\varepsilon_i = \varepsilon/\sqrt{t}$ and find the corresponding N_i. Then if $J = \max(N_1, N_2, \ldots, N_t)$ we have for all $m \geq n \geq J$ that

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| = \sqrt{\left(\sum_{k=n}^{m} a_{1,k}\right)^2 + \cdots + \left(\sum_{k=n}^{m} a_{t,k}\right)^2} \leq \sqrt{\varepsilon^2} = \varepsilon.$$

Conversely, let each $\varepsilon_i > 0$ be given and choose N such that

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leq \min(\varepsilon_1, \ldots, \varepsilon_t) \quad \text{for } m \geq n \geq N.$$

Then for each i and for all $m \geq n \geq N$ we have

$$\left|\sum_{k=n}^{m} a_{i,k}\right| = \sqrt{\left(\sum_{k=n}^{m} a_{i,k}\right)^2} \leq \sqrt{\left(\sum_{k=n}^{m} a_{1,k}\right)^2 + \cdots + \left(\sum_{k=n}^{m} a_{t,k}\right)^2} \leq \varepsilon_i.$$

Therefore (3.4) holds for each i and the proof is complete. □

Theorem. *If* $\sum \mathbf{a}_n$ *converges, then* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$.

Proof. This is immediate from Theorem 3.4 combined with the original Theorem 3.23. □

Theorem. *If* $|\mathbf{a}_n| \leq c_n$ *for* $n \geq N_0$*, where* N_0 *is some fixed integer, and if* $\sum c_n$ *converges, then* $\sum \mathbf{a}_n$ *converges.*

Proof. The proof is the same as the original proof of Theorem 3.25: given $\varepsilon > 0$, by the Cauchy criterion there exists $N \geq N_0$ such that $m \geq n \geq N$ implies

$$\sum_{k=n}^{m} c_k \leq \varepsilon.$$

By the triangle inequality,

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leq \sum_{k=n}^{m} |\mathbf{a}_k| \leq \sum_{k=n}^{m} c_k \leq \varepsilon.$$ □

Theorem (Root Test)**.** *Given* $\sum \mathbf{a}_n$*, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|\mathbf{a}_n|}.$$

Then

(*a*) *if* $\alpha < 1$, $\sum \mathbf{a}_n$ *converges;*

(*b*) *if* $\alpha > 1$, $\sum \mathbf{a}_n$ *diverges;*

(*c*) *if* $\alpha = 1$, *the test gives no information.*

Proof. Again, this proof is an easy generalization of the original proof of Theorem 3.33:

If $\alpha < 1$, choose β so that $\alpha < \beta < 1$. Then we can find an integer N such that

$$\sqrt[n]{|\mathbf{a}_n|} < \beta \quad \text{for } n \geq N.$$

Then $|\mathbf{a}_n| < \beta^n$ for all $n \geq N$, and the convergence of $\sum \mathbf{a}_n$ follows from the comparison test, since $\sum \beta^n$ converges.

If $\alpha > 1$ then there is a sequence $\{n_k\}$ such that $\sqrt[n]{|\mathbf{a}_{n_k}|} \to \alpha$. Hence $|\mathbf{a}_n| > 1$ for infinitely many values of n and we cannot have $\mathbf{a}_n \to \mathbf{0}$. Therefore $\sum \mathbf{a}_n$ diverges.

The fact that $\sum 1/n$ diverges while $\sum 1/n^2$ converges shows that $\alpha = 1$ gives no information about convergence. □

Theorem (Ratio Test)**.** *The series* $\sum \mathbf{a}_n$

(*a*) *converges if*

$$\limsup_{n\to\infty} \frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < 1,$$

(*b*) *diverges if*

$$\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} \geq 1 \quad \text{for all } n \geq n_0,$$

where n_0 *is some fixed integer.*

Proof. If the first condition holds, then we can find $\beta < 1$ and an integer N such that

$$\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < \beta \quad \text{for } n \geq N.$$

Then

$$\begin{aligned} |\mathbf{a}_{N+1}| &< \beta|\mathbf{a}_N|, \\ |\mathbf{a}_{N+2}| &< \beta|\mathbf{a}_{N+1}| < \beta^2|\mathbf{a}_N|, \\ &\vdots \\ |\mathbf{a}_{N+p}| &< \beta^p|\mathbf{a}_N|. \end{aligned}$$

So for $n \geq N$,

$$|\mathbf{a}_n| < |\mathbf{a}_N|\beta^{-N}\beta^n,$$

and convergence follows from the comparison test since $\sum \beta^n$ converges.

In the case where $|\mathbf{a}_{n+1}| \geq |\mathbf{a}_n|$ for $n \geq n_0$, it is clear that $\mathbf{a}_n$ does not tend to $\mathbf{0}$ and $\sum \mathbf{a}_n$ diverges. □

Theorem. *Suppose*

(*a*) *the partial sums* $\mathbf{A}_n$ *of* $\sum \mathbf{a}_n$ *form a bounded sequence;*

(*b*) $b_0 \geq b_1 \geq b_2 \geq \cdots$*;*

(*c*) $\lim_{n\to\infty} b_n = 0$.

Then $\sum b_n \mathbf{a}_n$ *converges.*

Proof. Since $\{\mathbf{A}_n\}$ is bounded we can find M such that $|\mathbf{A}_n| \leq M$ for all n. Given $\varepsilon > 0$ there is an integer N such that $b_N \leq \varepsilon/2M$. For $N \leq p \leq q$, we have

$$\begin{aligned} \left|\sum_{n=p}^{q} b_n\mathbf{a}_n\right| &= \left|\sum_{n=p}^{q}(b_n - b_{n+1})\mathbf{A}_n + b_q\mathbf{A}_q - b_p\mathbf{A}_{p-1}\right| \\ &\leq M\left|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2Mb_p \leq 2Mb_N \leq \varepsilon. \end{aligned}$$

Therefore $\sum b_n\mathbf{a}_n$ converges by the Cauchy criterion. □

Theorem. *If $\sum \mathbf{a}_n$ converges absolutely, then $\sum \mathbf{a}_n$ converges.*

Proof. From the triangle inequality, we have

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leq \sum_{k=n}^{m} |\mathbf{a}_k|,$$

so the series $\sum \mathbf{a}_n$ converges by the Cauchy criterion. □

Theorem. *If $\sum \mathbf{a}_n = \mathbf{A}$, and $\sum \mathbf{b}_n = \mathbf{B}$, then $\sum(\mathbf{a}_n + \mathbf{b}_n) = \mathbf{A} + \mathbf{B}$, and $\sum c\mathbf{a}_n = c\mathbf{A}$, for any fixed c.*

Proof. For each $n \geq 0$, set

$$\mathbf{A}_n = \sum_{k=0}^{n} \mathbf{a}_k \quad \text{and} \quad \mathbf{B}_n = \sum_{k=0}^{n} \mathbf{b}_k.$$

Then

$$\mathbf{A}_n + \mathbf{B}_n = \sum_{k=0}^{n} (\mathbf{a}_k + \mathbf{b}_k) \quad \text{and} \quad c\mathbf{A}_n = \sum_{k=0}^{n} c\mathbf{a}_n.$$

So

$$\lim_{n\to\infty} (\mathbf{A}_n + \mathbf{B}_n) = \lim_{n\to\infty} \mathbf{A}_n + \lim_{n\to\infty} \mathbf{B}_n = \mathbf{A} + \mathbf{B}$$

and

$$\lim_{n\to\infty} c\mathbf{A}_n = c\mathbf{A}.$$ □

Theorem. *If $\sum \mathbf{a}_n$ is a series of vectors which converges absolutely, then every rearrangement of $\sum \mathbf{a}_n$ converges, and they all converge to the same sum.*

Proof. Again, the proof is mostly identical:

Let $\sum \mathbf{a}'_n$ be a rearrangement, with $\mathbf{a}'_n = \mathbf{a}_{k_n}$ and with partial sums $\mathbf{s}'_n$. Given $\varepsilon > 0$, there exists an integer N such that

$$\sum_{i=n}^{m} |\mathbf{a}_i| \leq \varepsilon \quad \text{for all } m \geq n \geq N.$$

Choose p such that the integers $1, 2, \ldots, N$ are all contained within the set $k_1, k_2, \ldots, k_p$ (where each k_i is defined as in Definition 3.52). Then if $n > p$, the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ will cancel in the difference $\mathbf{s}_n - \mathbf{s}'_n$ so that $|\mathbf{s}_n - \mathbf{s}'_n| \leq \varepsilon$. Hence $\{\mathbf{s}'_n\}$ converges to the same sum as $\{\mathbf{s}_n\}$. □

3.16 Exercise 16

Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(*a*) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

Proof. Since $x_1 > \sqrt{\alpha}$, we have $\alpha < x_1^2$. Note also that for each $n \geq 1$,

$$\begin{aligned} x_{n+1}^2 - \alpha &= \frac{1}{4}\left(x_n + \frac{\alpha}{x_n}\right)^2 - \frac{1}{4}\left(\frac{4x_n\alpha}{x_n}\right) \\ &= \frac{1}{4}\left(x_n - \frac{\alpha}{x_n}\right)^2 \geq 0. \end{aligned}$$

Therefore $\alpha < x_n^2$ for all n. So for each n,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \leq x_n,$$

and we see that $\{x_n\}$ is a monotonically decreasing sequence.
Now, since $\{x_n\}$ is a monotonic sequence that is bounded below by $\sqrt{\alpha}$, it must converge to some value x. Then we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right).$$

So $2x = x + \alpha/x$ hence $x = \sqrt{\alpha}$. □

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \dots).$$

Proof. If $\varepsilon_n = x_n - \sqrt{\alpha}$ then we have

$$\begin{aligned} \varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\ &= \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}. \end{aligned}$$

Set $\beta = 2\sqrt{\alpha}$. Then a simple induction argument will show that

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad \text{for } n \geq 1.$$ □

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < 1/10$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Proof. Since $\sqrt{3} < 9/5$ we have

$$\frac{\varepsilon_1}{\beta} = \frac{x_1 - \sqrt{\alpha}}{2\sqrt{\alpha}} = \frac{2 - \sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}}{3} - \frac{1}{2} < \frac{3}{5} - \frac{1}{2} = \frac{1}{10}.$$

Then we have

$$\varepsilon_5 < \beta\left(\frac{1}{10}\right)^{16} < 4 \cdot 10^{-16} \quad \text{and} \quad \varepsilon_6 < \beta\left(\frac{1}{10}\right)^{32} < 4 \cdot 10^{-32}.$$ □

3.19 Exercise 19

Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Proof. $x(a)$ is the set of all real numbers in $[0, 1]$ which can be written in ternary without using the digit 1. But in removing the middle third of the interval $[0, 1]$, we are removing those numbers whose ternary expansion must have a 1 as the first digit (note that while $1/3 = 0.1_3$, we can also write $1/3 = 0.0\bar{2}_3$).

And in removing the middle third of the intervals $[0, 1/3]$ and $[2/3, 1]$ we are removing those numbers whose ternary expansion has a 1 in the second digit. This process continues, so that at each step we are removing numbers whose ternary expansions have a 1 in the nth place. This gives an intuitive explanation for why these two sets are the same. □

3.20 Exercise 20

Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Proof. Given $\varepsilon > 0$, we can find N_0 such that

$$d(p_{n_i}, p) < \frac{\varepsilon}{2} \quad \text{for all } n_i \geq N_0.$$

Since $\{p_n\}$ is a Cauchy sequence, we can also find N_1 such that

$$d(p_m, p_n) < \frac{\varepsilon}{2} \quad \text{for all } m, n \geq N_1.$$

Taking $N = \max(N_0, N_1)$ we have

$$d(p_m, p) \leq d(p_m, p_{n_i}) + d(p_{n_i}, p) < \varepsilon \quad \text{for all } m, n_i \geq N.$$

Hence $p_n \to p$. □

3.21 Exercise 21

Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Proof. For each n, choose $x_n \in E_n$. Then $\{x_n\}$ is a Cauchy sequence since $E_{n+1} \subset E_n$ and $\operatorname{diam} E_n \to 0$. Therefore the sequence $\{x_n\}$ must converge since X is complete. Suppose it converges to x. Then x is a limit point of E_n for each n, so $x \in E_n$ since E_n is closed. Therefore $x \in \bigcap E_n$. And if $y \in \bigcap E_n$, the fact that $\operatorname{diam} E_n \to 0$ means that $d(x, y) < \varepsilon$ for all $\varepsilon > 0$, so that in fact $x = y$. □

3.22 Exercise 22

Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X).

Proof. Pick $x_1 \in G_1$. Since G_1 is open, there is a neighborhood of x_1 contained entirely within G_1. Make the neighborhood small enough so that its closure is contained within G_1. That is, we find a closed ball E_1 centered at x_1 with radius r_1 such that $E_1 \subset G_1$.

Assuming that the closed ball E_n centered at x_n with radius r_n has been chosen so that $E_n \subset G_n$, pick a point $y \in E_n$. Since G_{n+1} is dense in X, either $y \in G_{n+1}$ or y is a limit point of G_{n+1}. If the former, set $x_{n+1} = y$. If the latter, take a neighborhood of y with radius small enough so that the neighborhood is contained within E_n. Then this neighborhood must contain points of G_{n+1}, so choose one and call it x_{n+1}. Now take a closed ball E_{n+1} centered at x_{n+1} and make its radius r_{n+1} small enough so that it is contained within $E_n \cap G_{n+1}$.

By construction, each E_n is a closed nonempty and bounded set with $E_1 \supset E_2 \supset E_3 \supset \cdots$. Moreover, since $r_n \to 0$ we have $\lim \operatorname{diam} E_n = 0$ as well. Since X is complete, we know by the previous exercise that $\bigcap_1^\infty E_n$ is nonempty. But each $E_n \subset G_n$, so this implies that $\bigcap_1^\infty G_n$ is nonempty as well. □

3.23 Exercise 23

Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Proof. Given $\varepsilon > 0$ choose N large enough so that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \quad \text{and} \quad d(q_n, q_m) < \frac{\varepsilon}{2} \quad \text{for all } m, n \geq N.$$

Since for any m, n

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n),$$

we have for $m, n \geq N$ that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \varepsilon.$$

Therefore the sequence $\{d(p_n, q_n)\}$ is Cauchy. Any Cauchy sequence of real numbers must converge (R^1 is complete), so $\{d(p_n, q_n)\}$ converges. □

3.24 Exercise 24

Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

Proof. Denote the relation by $\sim$. Clearly $\{p_n\}$ is equivalent to itself, since $d(p_n, p_n) = 0$ for all n, so $\sim$ is reflexive. Since $d(p_n, q_n) = d(q_n, p_n)$ we also have that $\sim$ is symmetric.

Now suppose that $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ are sequences with $\{p_n\}$ equivalent to $\{q_n\}$ and $\{q_n\}$ equivalent to $\{r_n\}$. Since

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n),$$

it follows that

$$\lim_{n\to\infty} d(p_n, r_n) = 0,$$

so $\{p_n\}$ is equivalent to $\{r_n\}$ and $\sim$ is transitive. This shows that $\sim$ is an equivalence relation. □

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

Proof. Suppose $\{p'_n\}$ is any sequence equivalent to $\{p_n\}$ and $\{q'_n\}$ is any sequence equivalent to $\{q_n\}$. Then

$$d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$$

so

$$|d(p_n, q_n) - d(p'_n, q'_n)| \leq d(p_n, p'_n) + d(q'_n, q_n).$$

Since the right-hand side of this inequality can be made arbitrarily small by choosing large enough n, it follows that

$$\lim_{n\to\infty} (d(p_n, q_n) - d(p'_n, q'_n)) = 0$$

or

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n).$$

Therefore Δ is a well-defined function from $X^* \times X^*$ into R^1.

It is evident that $\Delta(P, P) = 0$ and $\Delta(P, Q) = \Delta(Q, P)$. And the triangle inequality follows from the triangle inequality in X. Hence Δ is a distance function. □

(c) Prove that the resulting metric space X^* is complete.

Proof. Let $\{P_k\}$ be a Cauchy sequence in X^* and for each k, let $\{p_{k,n}\}$ be a Cauchy sequence in P_k. Choose a positive integer N_k such that

$$d(p_{k,n}, p_{k,m}) < \frac{1}{2^k} \quad \text{for all } m, n \geq N_k.$$

Now set $q_k = p_{k,N_k}$. Then

$$d(q_k, p_{k,n}) < \frac{1}{2^k} \quad \text{for all } n \geq N_k. \tag{3.5}$$

Since

$$\Delta(P_i, P_j) = \lim_{n\to\infty} d(p_{i,n}, p_{j,n}),$$

we can also find $M(i, j)$ so that

$$|\Delta(P_i, P_j) - d(p_{i,n}, p_{j,n})| < \frac{1}{2^{\max(i,j)}} \quad \text{for all } n \geq M(i, j).$$

Now, for any i, j, n, we have

$$d(q_i, q_j) \leq d(q_i, p_{i,n}) + d(p_{i,n}, p_{j,n}) + d(p_{j,n}, q_j).$$

Therefore,

$$d(q_i, q_j) \leq \frac{1}{2^i} + \left(\Delta(P_i, P_j) + \frac{1}{2^{\max(i,j)}}\right) + \frac{1}{2^j}$$

for all $n \geq \max(N_i, N_j, M(i, j))$. Now $d(q_i, q_j)$ can be made arbitrarily small for sufficiently large i and j, so $\{q_k\}$ is a Cauchy sequence.

Let P be the equivalence class in X^* containing $\{q_k\}$. Then

$$\Delta(P_k, P) = \lim_{n\to\infty} d(p_{k,n}, q_n) \leq \lim_{n\to\infty} (d(p_{k,n}, q_k) + d(q_k, q_n)).$$

By (3.5) and the fact that $\{q_k\}$ is Cauchy, we have

$$\lim_{k\to\infty} P_k = P.$$

This shows that X^* is complete. □

(*d*) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

Proof. This result is immediate:

$$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p, q) = d(p, q).$$

Therefore φ preserves distance. □

(*e*) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Proof. Let $P \in X^*$. If $P \in \varphi(X)$ then we are done, so suppose $P \notin \varphi(X)$. Then we need to show that P is a limit point of $\varphi(X)$. Given $\varepsilon > 0$, let $\{p_n\} \in P$ and choose N so that

$$d(p_n, p_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

Then

$$\Delta(P, P_{p_N}) = \lim_{n \to \infty} d(p_n, p_N) \leq \varepsilon.$$

Since $P_{p_N} = \varphi(p_N)$, this shows that any neighborhood of P in X^* contains a point from $\varphi(X)$, so that $\varphi(X)$ is dense in X^*.

Lastly, if X is complete, then for each $P \in X^*$ and for each $\{p_n\} \in P$, $\{p_n\}$ must converge to a point p in X. Any sequence equivalent to $\{p_n\}$ will also converge to p, so $P = P_p = \varphi(p)$. Therefore $\varphi(X) = X^*$. □

3.25 Exercise 25

Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space?

Solution. Every Cauchy sequence in X converges to a unique real number, and two sequences in X are equivalent if and only if they converge to the same real number. So the completion of X is the real numbers. □

Chapter 4

Continuity

4.1 Exercise 1

Suppose f is a real function defined on R^1 which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0 \tag{4.1}$$

for every $x \in R^1$. Does this imply that f is continuous?

Solution. No. Consider the function $f\colon R^1 \to R^1$ given by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is an integer,}\\ 0, & \text{otherwise.}\end{cases}$$

This function clearly has infinitely many discontinuities.

Now fix a particular real number x. If x is an integer, then for any $\varepsilon > 0$ we may let $\delta = 1/2$, so that for all h with $0 < |h| < \delta$ we have

$$|f(x+h)-f(x-h)| = |0-0| = 0 < \varepsilon.$$

On the other hand, if x is not an integer, then let n be the integer nearest to x. For any $\varepsilon > 0$ take

$$\delta = \frac{|x-n|}{2}.$$

Again, for all h with $0 < |h| < \delta$ we have

$$|f(x+h)-f(x-h)| = |0-0| = 0 < \varepsilon.$$

We have shown that the function f satisfies (4.1) for any x in R^1, but f is not continuous. □

4.2 Exercise 2

If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof. Suppose $p \in f(\overline{E})$. If $p \in f(E)$ then certainly $p \in \overline{f(E)}$ and we are done. So suppose that $p \notin f(E)$. We want to show that p must be a limit point of $f(E)$.

By the choice of p there must be $q \in \overline{E}$ such that $p = f(q)$. But if $p \notin f(E)$ then $q \notin E$. Therefore q is a limit point of E.

Take any neighborhood N of p having radius $r > 0$. Since f is continuous, we may find $\delta > 0$ so that for any $x \in X$,

$$d(x, q) < \delta \quad \text{implies} \quad d(f(x), p) < r.$$

And q is a limit point of E, so we can certainly find $x \in E$ such that $d(x, q) < \delta$. Then $f(x) \in f(E)$ by definition, and we have $f(x) \neq p$ and $f(x) \in N$. Therefore p is indeed a limit point of $f(E)$ and the main proof is complete.

Finally, to see that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$, consider the continuous function $f\colon R \to R$ given by

$$f(x) = \frac{2}{x^2 + 1}.$$

If $E = (1, \infty)$ then $f(E) = (0, 1)$ so that $\overline{f(E)} = [0, 1]$. However $\overline{E} = [1, \infty)$ so $f(\overline{E}) = (0, 1]$. Then $f(\overline{E}) \subset \overline{f(E)}$ but equality does not hold. □

4.3 Exercise 3

Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof. Since the set $\{0\}$ is closed in R^1 and f is continuous, the corollary to Theorem 4.8 guarantees that $f^{-1}(\{0\})$ is closed in X. But $Z(f) = f^{-1}(\{0\})$ by definition, so $Z(f)$ is closed. □

4.4 Exercise 4

Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain).

Proof. Let $p \in f(X)$. Since $X = \overline{E}$, we have by Exercise 4.2 that $p \in \overline{f(E)}$ so that p is either in $f(E)$ or is a limit point of $f(E)$. Either way this shows that $f(E)$ is dense in $f(X)$.

Next, suppose $g(p) = f(p)$ for all $p \in E$ and let $q \in X - E$. Since E is dense in X, q is a limit point of E. Let $\varepsilon > 0$ be arbitrary. The continuity of f allows us to then find $\delta_1 > 0$ such that, for $x \in X$,

$$d(x, q) < \delta_1 \quad \text{implies} \quad d(f(x), f(q)) < \frac{\varepsilon}{2}.$$

Similarly, the continuity of g allows us to find $\delta_2 > 0$ such that

$$d(x, q) < \delta_2 \quad \text{implies} \quad d(g(x), g(q)) < \frac{\varepsilon}{2}.$$

Set $\delta = \min(\delta_1, \delta_2)$.

Pick $r \in E$ such that $d(q, r) < \delta$ (this is possible since q is a limit point of E). Since $g(r) = f(r)$, we then have

$$\begin{aligned} d(f(q), g(q)) &\leq d(f(q), f(r)) + d(f(r), g(q)) \\ &= d(f(q), f(r)) + d(g(r), g(q)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, we must have $d(f(q), g(q)) = 0$ so that $f(q) = g(q)$. □

4.5 Exercise 5

If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions.

Proof. Since E is closed, its complement E^c is open. Since E^c is an open set in R^1, we have by Exercise 2.29 that it is the union of an at most countable collection of disjoint segments. Let the nth segment be (a_n, b_n). Define the function $g\colon R^1 \to R^1$ as follows.

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \frac{1}{b_n - a_n}\big((b_n - x)f(a_n) + (x - a_n)f(b_n)\big) & \text{if } x \in (a_n, b_n). \end{cases}$$

Note that we are assuming that each segment (a_n, b_n) is bounded. If there is a segment of the form $(-\infty, b_n)$, then the proof will still work if we define $g(x)$ to be $f(b_n)$ for all x in this segment, and similarly for the segment (a_n, ∞).

Now, it is clear that g is continuous on the interior of E as well as on E^c. We need to show that g is continuous at the points a_n and b_n for $n = 1, 2, 3, \ldots$.

Fix a positive integer n. It is clear from the continuity of f that

$$g(a_n-) = f(a_n) \quad \text{and} \quad g(b_n+) = f(b_n).$$

Checking the limits on the inside, we find

$$g(a_n+) = f(a_n) + 0f(b_n) = f(a_n),$$

and

$$g(b_n-) = 0f(a_n) + f(b_n) = f(b_n).$$

So

$$\lim_{t \to a_n} g(t) = f(a_n) = g(a_n) \quad \text{and} \quad \lim_{t \to b_n} g(t) = f(b_n) = g(b_n),$$

which shows that g is continuous at a_n and b_n. It now follows that g is continuous on all of R^1.

To see that this result is not necessarily true if E is not closed, consider the function $f\colon E \to R^1$, where $E = R^1 - \{0\}$, given by

$$f(x) = \frac{1}{x}.$$

Then there is no way to define a function g so that $g(x) = f(x)$ on E with g being continuous at 0, since the left- and right-handed limits at 0 are not affected by the value of $g(0)$.

If $\mathbf{f}$ is vector-valued, then the same result (with essentially the same proof) holds if we define $g\colon R^1 \to R^k$ by

$$\mathbf{g}(x) = \begin{cases} \mathbf{f}(x) & \text{if } x \in E, \\ \frac{1}{b_n - a_n}\big((b_n - x)\mathbf{f}(a_n) + (x - a_n)\mathbf{f}(b_n)\big) & \text{if } x \in (a_n, b_n), \end{cases}$$

the only difference being that the values of f and g are in R^k rather than R^1. □

4.6 Exercise 6

If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.

Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof. Let $E \subset X$ and $f\colon E \to Y$, where X and Y are metric spaces. Let $G \subset X \times Y$ be the graph of f and let $g\colon E \to X \times Y$ be given by $g(x) = (x, f(x))$, so that $g(E) = G$.

Define the metric $d\colon (X \times Y)^2 \to R$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

It is easy to check that d is indeed a metric, so that $X \times Y$ becomes a metric space.

First, suppose f is continuous. We show that g is then continuous also. Fix a point $x \in E$. For any $\varepsilon > 0$, let $\delta_0 > 0$ be such that $d_Y(f(p), f(x)) < \varepsilon/2$ whenever $d_X(p, x) < \delta_0$. Now put

$$\delta = \min\left\{\delta_0, \frac{\varepsilon}{2}\right\}.$$

Then $d_X(p, x) < \delta$ implies

$$\begin{aligned} d(g(p), g(x)) &= d((p, f(p)), (x, f(x))) \\ &= d_X(p, x) + d_Y(f(p), f(x)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore g is continuous on E. The compactness of $G = g(E)$ now follows from Theorem 4.14.

Conversely, suppose G is compact, and assume that f is not continuous, say at a point $x \in E$. Set $\delta_n = 1/n$ for positive integers n. Since f is not continuous at x, there is an $\varepsilon > 0$ such that for each δ_n, we can pick an x_n so that $d_X(x_n, x) < \delta_n$ and

$$d_Y(f(x_n), f(x)) \geq \varepsilon. \tag{4.2}$$

Since G is compact, the sequence $\{(x_n, f(x_n))\}$ must have a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ that converges to a point $(y, f(y)) \in G$. Then $x_{n_k} \to y$. But

$x_n \to x$, so all subsequences converge to x also. Therefore $y = x$, and the subsequence $\{f(x_{n_k})\}$ converges to $f(x)$. But this is impossible since, by (4.2), we know that $d_Y(f(x_{n_k}), f(x)) \geq \varepsilon$. Therefore f must be continuous at x. And $x \in E$ was arbitrary, so f is continuous on E and the proof is complete. □

4.7 Exercise 7

If $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by $f(0,0) = g(0,0) = 0$, $f(x,y) = xy^2/(x^2+y^4)$, $g(x,y) = xy^2/(x^2+y^6)$ if $(x,y) \neq (0,0)$. Prove that f is bounded on R^2, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!

Proof. For all $(x,y) \in R^2$, we have

$$0 \leq (|x| - y^2)^2 = x^2 - 2|x|y^2 + y^4,$$

which implies that $|xy^2| \leq (x^2+y^4)/2$. So, when x and y are not both zero,

$$|f(x,y)| = \frac{|xy^2|}{x^2+y^4} \leq \frac{1}{2}.$$

So f is bounded on R^2.

Next, note that if $(x,y) \in R^2$ is such that $x = y^3$ and $x, y \neq 0$, then

$$g(x,y) = \frac{y^3(y^2)}{(y^3)^2+y^6} = \frac{y^5}{2y^6} = \frac{1}{2y}.$$

So, by choosing $y > 0$ small enough, we can always find a point (y^3, y) in any neighborhood of $(0,0)$ such that $g(y^3, y)$ is arbitrarily large. This shows that g is unbounded in every neighborhood of $(0,0)$.

To show that f is not continuous at $(0,0)$, note that in every neighborhood of $(0,0)$, we can find a point (x,y) such that $x = y^2$, so that if x and y are not both zero,

$$f(x,y) = \frac{y^2(y^2)}{(y^2)^2+y^4} = \frac{1}{2}.$$

Therefore, by choosing $\varepsilon < \frac{1}{2}$, there is no neighborhood of $(0,0)$ such that $|f(x,y) - f(0,0)| < \varepsilon$ for all (x,y) in the neighborhood. This shows that f is not continuous.

Finally, we consider restrictions of f and g to straight lines. Certainly f and g are continuous everywhere except $(0,0)$. So we only need to consider restrictions to straight lines which pass through $(0,0)$. Such lines have the form $ax + by = 0$. Fix any $a, b \in R$ and let f^* and g^* be the restrictions of f and g, respectively, to the line $ax + by = 0$.

If $b = 0$ then $x = 0$ for all y and we have $f^*(0,y) = g^*(0,y) = 0$ for all (x,y) on this line so both functions are continuous. On the other hand, if $b \neq 0$, then $y = -(a/b)x$ and we get

$$f^*(x, -ax/b) = \frac{(a^2/b^2)x^3}{x^2 + (a^4/b^4)x^4} = \frac{(ab)^2 x}{b^4 + a^4x^2},$$

and, similarly,

$$g^*(x, -ax/b) = \frac{(a^2/b^2)x^3}{x^2 + (a^6/b^6)x^6} = \frac{a^2b^4x}{b^6 + a^6x^4}.$$

Since the denominators of f^* and g^* are never zero, we conclude that both functions are continuous. □

4.8 Exercise 8

Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof. Let E and f be as stated. Since E is bounded, it is contained within some bounded interval $[a, b]$. And since f is uniformly continuous on E, we may choose $\delta > 0$ so that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$, for all $x, y \in E$. Pick any positive integer N larger than $(b - a)/\delta$ and define the $N + 1$ points $t_0, t_1, \ldots, t_N$ by

$$t_i = a + \frac{(b-a)i}{N} \quad \text{for } i = 0, 1, \ldots, N.$$

For each i with $1 \leq i \leq N$, define the interval $J_i = [t_{i-1}, t_i]$ and, if $J_i \cap E$ is nonempty, pick $x_i \in J_i \cap E$. Then let

$$M = \max_i \{|f(x_i)|\} + 1.$$

Now, for any $x \in E$, x belongs to some J_i, so

$$|x - x_i| \leq \frac{b - a}{N} < \delta,$$

which implies that $|f(x) - f(x_i)| < 1$. Therefore $|f(x)| \leq M$ and f is bounded.

Note that the boundedness of E was necessary, since otherwise the identity function on R^1 would serve as a counterexample. □

4.9 Exercise 9

Show that the requirement in the definition of uniform continuity can be restated as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$.

Proof. By definition, $\operatorname{diam} E < \delta$ if and only if $d(x, y) < \delta$ for all $x, y \in E$, and $\operatorname{diam} f(E) < \varepsilon$ if and only if $d(f(x), f(y)) < \varepsilon$ for all $x, y \in E$.

If $f\colon X \to Y$ is uniformly continuous, then for all $\varepsilon > 0$ there is $\delta > 0$ so that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$. If $E \subset X$ is any subset such that $\operatorname{diam} E < \delta$, then $d(x, y) < \delta$ for all $x, y \in E$ so $d(f(x), f(y)) < \varepsilon$ which shows that $\operatorname{diam} f(E) < \varepsilon$.

Conversely, suppose f is such that to every $\varepsilon > 0$ there is $\delta > 0$ so that $\operatorname{diam} E < \delta$ implies $\operatorname{diam} f(E) < \varepsilon$. Given any $\varepsilon > 0$, choose such a $\delta > 0$. For any $x, y \in X$ with $d(x, y) < \delta$, define $E = \{x, y\}$ so that $\operatorname{diam} E < \delta$. Then $\operatorname{diam} f(E) < \varepsilon$ which means $d(f(x), f(y)) < \varepsilon$ as required. □

4.10 Exercise 10

Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

Theorem (4.19)**.** *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Proof. Put $\delta_n = 1/n$. If f is not uniformly continuous, then for some $\varepsilon > 0$ there are points p_n and q_n with $d_X(p_n, q_n) < \delta_n$ but $d_Y(f(p_n), f(q_n)) \geq \varepsilon$ for all n. Then $\{p_n\}$ and $\{q_n\}$ are sequences in X. Since X is compact, there is some subsequence $\{p_{n_k}\}$ and some subsequence $\{q_{n_k}\}$ such that $p_{n_k} \to p$ and $q_{n_k} \to q$ for some $p, q \in X$.

Since $d_X(p_n, q_n) \to 0$, we must have $p = q$. Since f is continuous, we also have $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(q)$. Choose δ_1 so that $d(f(p_{n_k}), f(p)) < \varepsilon/2$ for all $k \geq N_1$ and choose δ_2 so that $d(f(q_{n_k}), f(q)) < \varepsilon/2$ for all $k \geq N_2$. Then for all $k \geq \max\{N_1, N_2\}$, we have

$$d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(p)) + d(f(p), f(q_{n_k})) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This contradicts our earlier assertion. Therefore f must be uniformly continuous on X. □

4.11 Exercise 11

Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

Proof. Fix any $\varepsilon > 0$ and let $\{x_n\}$ be a Cauchy sequence in X. Since f is uniformly continuous, there is a $\delta > 0$ such that

$$d(f(x), f(y)) < \varepsilon \quad \text{whenever} \quad d(x, y) < \delta. \tag{4.3}$$

Since $\{x_n\}$ is Cauchy, we can find a positive integer N such that $d(x_m, x_n) < \delta$ whenever $m, n \geq N$. Then, by (4.3), we have $d(f(x_m), f(x_n)) < \varepsilon$ for all $m, n \geq N$, and we see that the sequence $\{f(x_n)\}$ is Cauchy in Y.

Now, let E be a dense subset of X and suppose $f\colon E \to R^1$ is uniformly continuous. We want to show that f has a continuous extension from E to X. Pick any point $p \in X$. Then either $p \in E$ or p is a limit point of E. If the former, define $g(p) = f(p)$.

If p is a limit point, then there is a sequence $\{p_n\}$ in E that converges to p. Since every convergent sequence in a metric space is Cauchy, $\{p_n\}$ is a Cauchy sequence, so our previous result guarantees that $\{f(p_n)\}$ is also Cauchy, and hence converges to a point q in R^1. Define $g(p) = q$ in this case. It is then easy to check that g is a continuous function from X to R^1. □

4.12 Exercise 12

A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

Solution. Let X, Y, Z be metric spaces, let $f\colon X \to Y$ and $g\colon f(X) \to Z$ be uniformly continuous functions. Then the composite function $g \circ f$ is uniformly continuous.

To prove this, let $\varepsilon > 0$. By the uniform continuity of g, there is $\eta > 0$ such that $d(g(p), g(q)) < \varepsilon$ whenever $d(p, q) < \eta$, for any $p, q \in f(X)$. Also, by the uniform continuity of f, we can find $\delta > 0$ such that $d(f(s), f(t)) < \eta$ for any $s, t \in X$ such that $d(s, t) < \delta$.

Combining these two results, we have the following: given any $x, y \in X$ with $d(x, y) < \delta$, we have

$$d(g(f(x)), g(f(y))) < \varepsilon.$$

Therefore $g \circ f$ is uniformly continuous. □

4.13 Exercise 13

Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X. (Uniqueness follows from Exercise 4.4.)

Could the range space R^1 be replaced by R^k? By any compact metric space? By any complete metric space? By any metric space?

Solution. Let $f\colon E \to R$ be uniformly continuous, where E is dense in X. For each point p of X and each positive integer n, define $V_n(p)$ to be the set of all $q \in E$ with $d(p, q) < 1/n$. Define the set

$$F(p) = \bigcap_{n=1}^{\infty} \overline{f(V_n(p))}.$$

Then $F(p)$ is the intersection of a nested set of closed and bounded (and hence compact) subsets of R^1, so we know by the Corollary to Theorem 2.36 that $F(p)$ is nonempty.

Since f is uniformly continuous and since $\operatorname{diam} V_n(p) \to 0$, we know by Exercise 4.9 that $\operatorname{diam} f(V_n(p)) \to 0$ also. Then by Theorem 3.10 (b), we know $F(p)$ consists of exactly one point. Call this point $g(p)$. Note that if $p \in E$, then $g(p) = f(p)$.

To complete the proof, we must show that the function g is continuous. Let $\varepsilon > 0$. By the uniform continuity of f, we can find $\delta > 0$ so that

$$|f(p) - f(q)| < \frac{\varepsilon}{3} \quad \text{for all } p, q \in E \text{ with } d(p, q) < \delta.$$

Now let $x, y \in X$ with $d(x, y) < \delta$. Choose $p \in E$ such that

$$d(x, p) < \frac{\delta - d(x, y)}{2} \quad \text{and} \quad |f(p) - g(x)| < \frac{\varepsilon}{3}$$

(this is possible by our construction above). Similarly, choose $q \in E$ such that

$$d(y,q) < \frac{\delta - d(x,y)}{2} \quad \text{and} \quad |f(q) - g(y)| < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned} d(p,q) &\leq d(p,x) + d(x,y) + d(y,q) \\ &< \frac{\delta - d(x,y)}{2} + d(x,y) + \frac{\delta - d(x,y)}{2} \\ &= \delta, \end{aligned}$$

so

$$\begin{aligned} |g(x) - g(y)| &\leq |g(x) - f(p)| + |f(p) - f(q)| + |f(q) - g(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

So g is uniformly continuous on X.

This proof relied on the property that closed and bounded subsets of R^1 are compact. The same proof would work in R^k as well as any compact metric space. In fact, this result holds in any complete metric space, as we can see from the proof in Exercise 4.11.

Completeness is necessary, however: Let X be any metric space that is not complete (for a concrete example, we could take the rationals Q). Take the identity function $f\colon X \to X$ given by $f(p) = p$. f is certainly uniformly continuous. If X^* is the completion of X (see Exercise 3.24), then there is no continuous extension of f from X to X^* (having the same codomain X). For, if there is such an extension, let it be $g\colon X^* \to X$. Now let $h\colon X^* \to X^*$ be the identity function on X^*. Then h is also a continuous extension of f, but with a codomain of X^* instead of X. So we have found two different continuous extensions of f to X^*, and this contradicts what we know from Exercise 4.4. □

4.14 Exercise 14

Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof. Define $g\colon [0,1] \to R$ by $g(x) = f(x) - x$. Then g is clearly continuous. If $g(0) = 0$ or $g(1) = 0$ then we are done, so suppose $g(0)$ and $g(1)$ are both nonzero. Then $g(0) > 0$ and $g(1) < 0$, so by Theorem 2.43, there is a point c in $(0,1)$ such that $g(c) = 0$. Then $f(c) = c$ as required. □

4.15 Exercise 15

Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof. Suppose f is a continuous mapping from R^1 into R^1 that is not monotonic. We will show that f cannot be open. Because f is not monotonic, we may choose points $a, b, c \in R^1$ with $a < b < c$ such that

$$f(a) < f(b) \quad \text{and} \quad f(b) > f(c)$$

(the inequalities could be reversed, but then we could just consider the function $-f$, so there is no loss of generality). Now $[a, c]$ is a closed bounded set in R^1, hence it is compact. By the continuity of f and by Theorem 4.16, we know that f attains a maximum value at a point $x \in (a, c)$.

Now consider the set $f(I)$ where I is the segment (a, c). I is clearly open. However, $f(I)$ cannot be open, for $f(x)$ is not an interior point of $f(I)$ ($f(x)$ is the supremum of the set, so any neighborhood of $f(x)$ must contain points outside of $f(I)$). Therefore f is not an open map. The result now follows by the contrapositive. □

4.16 Exercise 16

Let $[x]$ denote the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

Solution. Clearly both functions are discontinuous at all integer values of x and continuous everywhere else. If n is an integer, then $[n-] = n - 1$ and $[n+] = n$, so all discontinuities of $[x]$ are simple discontinuities. Similarly, $(n-) = 1$ and $(n+) = 0$, and so (x) also has only simple discontinuities. □

4.17 Exercise 17

Let f be a real function defined on (a, b) Prove that the set of points at which f has a simple discontinuity is at most countable.

Proof. Let E be the set of points x such that $f(x-) < f(x+)$ and fix any $x \in E$. Since the rationals are dense in R^1, we can choose a rational p with $f(x-) < p < f(x+)$. Now since $f(x-)$ exists and is less than p, we may take $\varepsilon = p - f(x-)$ and choose $\delta > 0$ so that $f(t) < p$ whenever $t \in (x - \delta, x)$. We may find a rational number q with $x - \delta < q < x$, so that $f(t) < p$ for all $t \in (q, x)$. Similarly, we may find a rational number $r > x$ with $f(t) > p$ for all $t \in (x, r)$. Then x is described by the ordered triple (p, q, r) of rational numbers.

Now suppose x and y are members of E that are described by the same triple (p, q, r). Since $f(t) < p$ for all $t \in (q, x)$, we must have $y \geq x$ since otherwise $f(y+) > p$ would be a contradiction. Similarly, since $f(t) > p$ for all $t \in (x, r)$, we must have $y \leq x$. It follows that $y = x$ so that the two discontinuities are actually the same. Therefore every triple (p, q, r) corresponds to at most one member of E. Since the set of triples of rational numbers is countable, we know that E is at most countable.

By defining the function g on (a, b) by $g(x) = -f(x)$, we see that any simple discontinuity of f such that $f(x-) > f(x+)$ must give $g(x-) < g(x+)$ so that the above result shows that such discontinuities are also countable.

The last type of simple discontinuity we have to consider is where $f(x-) = f(x+)$, so that $f(x) \neq f(x-)$. If E is the set of such discontinuities, then we may use the same ordered triple to describe E, except that we may choose p to be between $f(x-)$ and $f(x)$, with q and r such that either $f(t) < f(x)$ for all $t \in (q, x) \cup (x, r)$ or $f(t) > f(x)$ for all $t \in (q, x) \cup (x, r)$. Again, we find that each triple corresponds to at most one member of E, so the set is at most countable.

Combining these results, we see that the set of all simple discontinuities of f is at most countable. □

4.18 Exercise 18

Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & \left(x = \frac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof. Choose any $\varepsilon > 0$. First, let x be any irrational number. We can find a positive integer N such that $1/N < \varepsilon$. Now for each positive integer $n \leq N$, take the largest integer k_n that is less than nx. Then we have

$$\frac{k_n}{n} < x < \frac{k_n + 1}{n}, \quad n = 1, 2, \ldots, N.$$

Now for each positive $n \leq N$, set

$$\delta_n = \min\left\{\left|x - \frac{k_n}{n}\right|, \left|x - \frac{k_n + 1}{n}\right|\right\},$$

and put

$$\delta = \min_{1 \leq i \leq N} \{\delta_i\}.$$

Then for any rational $y = p/q$ (in lowest terms) such that $|x - y| < \delta$, we must have $q > N$ so that

$$|f(y) - f(x)| = \frac{1}{q} < \varepsilon.$$

On the other hand, if y is irrational, then $|f(y) - f(x)| = 0 < \varepsilon$. This shows that f is continuous at any irrational x.

Now if x is rational, let it be written $x = p/q$ in lowest terms, so that $f(x) = 1/q$. We can perform the same construction as before, except that we now have the possibility that nx is an integer, so we want to only choose n such that $q < n \leq N$, where N is chosen to be at least as large as $q + 1$. Then the same proof as before will show that

$$\lim_{y \to x} f(y) = 0 \neq f(x).$$

So f is not continuous at each rational x, but the limit does exist, so these discontinuities are all simple. □

4.19 Exercise 19

Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed.

Prove that f is continuous.

Proof. Suppose to the contrary that f is not continuous, say at the point x_0. Then for some $\varepsilon > 0$ we may find a sequence of points $\{x_n\}$ which converges to x_0 but such that $f(x_n) - f(x_0) \geq \varepsilon$ for all n (if this is not possible, then there is a sequence with $f(x_0) - f(x_n) \geq \varepsilon$ and a similar argument will work). Choose a rational number $r \in (f(x_0), f(x_0) + \varepsilon)$.

Then for each n, r is between $f(x_0)$ and $f(x_n)$. Since f has the intermediate value property, we may find t_n between x_0 and x_n such that $f(t_n) = r$. Since $x_n \to x_0$ we also have $t_n \to x_0$. This shows that x_0 is a limit point of the set of all t such that $f(t) = r$. But $f(x_0) \neq r$, so this contradicts the fact that the set is closed. Therefore f must be continuous. □

4.20 Exercise 20

If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(*a*) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

Proof. Suppose $\rho_E(x) = 0$. If $x \in E$ then certainly $x \in \overline{E}$, so suppose $x \notin E$. Then there is a $z_n \in E$ such that $d(x, z_n) < 1/n$ for each positive integer n. Therefore x is a limit point of E so $x \in \overline{E}$.

Conversely, if $x \in \overline{E}$ then either $x \in E$ or x is a limit point of E. If $x \in E$ then certainly $\rho_E(x) = 0$. If x is a limit point, then we can find a sequence $\{z_n\}$ of points in E with $z_n \to x$. This implies that $d(x, z_n) \to 0$, so $\rho_E(x) = 0$ in this case also. □

(*b*) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$, $y \in X$.

Proof. Let z be any point in E. Then

$$\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).$$

This holds for all $z \in E$, and by taking the infimum over z, the inequality becomes

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

Therefore

$$\rho_E(x) - \rho_E(y) \leq d(x, y).$$

In a similar way, we can show that

$$\rho_E(y) - \rho_E(x) \le d(y,x) = d(x,y).$$

Therefore

$$|\rho_E(x) - \rho_E(y)| \le d(x,y) \quad \text{for all } x, y \in X.$$

This shows that ρ_E is uniformly continuous on X. □

4.21 Exercise 21

Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Proof. If $\rho_F(p) = 0$ for some $p \in K$, we know by the previous exercise (Exercise 4.20) that $p \in \overline{F}$. But F is closed, so $F = \overline{F}$ and we would have $p \in K \cap F$, which contradicts the fact that the sets are disjoint. So $\rho_F(p)$ is strictly positive.

Since ρ_F is a continuous function on the compact metric space K, we also know by Theorem 4.16 that ρ_F attains a minimum value on K. Thus we can find a $\delta > 0$ such that $\rho_F(p) > \delta$ for all $p \in K$. Restated, this means that $d(p,q) > \delta$ for all $p \in K$ and $q \in F$.

If F and K are closed but not compact, the conclusion may fail. Take for example the closed sets in R^2 given by

$$F = \{(x,0) \in R^2 \mid x \in R\} \quad \text{and} \quad K = \{(x,y) \in R^2 \mid y = 1/(1+x^2)\}.$$

Then $\rho_F(p)$ is still positive for all $p \in K$, but for any $\delta > 0$ we can always find a point p in K and q in F with $d(p,q) < \delta$. □

4.22 Exercise 22

Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0,1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 4.3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}\left(\left[0, \tfrac{1}{2}\right)\right), \quad W = f^{-1}\left(\left(\tfrac{1}{2}, 1\right]\right),$$

show that V and W are open and disjoint, and that $A \subset V, B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

Proof. We have already seen in Exercise 4.20 that ρ_A and ρ_B are continuous. The continuity of f then follows from Theorem 4.4 (note that the denominator is nonzero since $\rho_A(p)$ and $\rho_B(p)$ cannot be simultaneously zero).

We also know that, since A and B are closed, $\rho_A(p) = 0$ if and only if $p \in A$, and $\rho_B(p) = 0$ if and only if $p \in B$. This implies that $f(p) = 0$ if and only if $p \in A$ and $f(p) = 1$ if and only if $p \in B$, and that $f(X)$ lies within the interval $[0, 1]$.

Now let V and W be as described above. Take any point $p \in V$. Suppose $\rho_A(p) = r < \frac{1}{2}$. Then if we choose any positive $\delta < \frac{1}{2} - r$, we see that the δ-neighborhood of p is entirely contained within V. Therefore V is open, and certainly $A \subset V$. In an entirely similar way we can show that W is open and that $B \subset W$. Lastly, V and W must be disjoint since they are preimages of disjoint sets under the same function. □

4.25 Exercise 25

If $A \subset R^k$ and $B \subset R^k$ define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(*a*) If K is compact and C is closed in R^k, prove that $K + C$ is closed.

Proof. Take $\mathbf{z} \notin K + C$ and let

$$F = \mathbf{z} - C = \{\mathbf{z} - \mathbf{y} \mid \mathbf{y} \in C\}.$$

Now suppose $\mathbf{a} \in F \cap K$. Then $\mathbf{a} = \mathbf{z} - \mathbf{c}$ for some $\mathbf{c} \in C$, so that $\mathbf{z} = \mathbf{a} + \mathbf{c} \in K + C$, which contradicts our choice of $\mathbf{z}$. Therefore F and K are disjoint sets.

Next, we show that F is closed. Let $\mathbf{a}$ be a limit point of F. Let $\{\mathbf{f}_n\}$ be a sequence of points in F converging to $\mathbf{a}$. Then each $\mathbf{f}_n$ can be written $\mathbf{f}_n = \mathbf{z} - \mathbf{c}_n$ for $\mathbf{c}_n \in C$. Then $\{\mathbf{c}_n\}$ is a sequence which converges to a point $\mathbf{c}$ in C (since C is closed). Then $\mathbf{a} = \mathbf{z} - \mathbf{c}$ so $\mathbf{a} \in F$.

Since K is compact and F is closed, with K and F disjoint, we can, by Exercise 4.21, find $\delta > 0$ such that $|\mathbf{x} - \mathbf{y}| > \delta$ for any $\mathbf{x} \in K$ and $\mathbf{y} \in F$. We will show that the open ball with center $\mathbf{z}$ and radius δ can have no points in common with $K + C$. If there is such a point let it be $\mathbf{x}$. Then we can write $\mathbf{x} = \mathbf{k} + \mathbf{c}$ for $\mathbf{k} \in K$ and $\mathbf{c} \in C$. But then

$$|\mathbf{k} - (\mathbf{z} - \mathbf{c})| = |\mathbf{x} - \mathbf{z}| < \delta,$$

which is a contradiction since $\mathbf{k} \in K$ and $\mathbf{z} - \mathbf{c} \in F$. This contradiction shows that $(K + C)^c$ is open, so $K + C$ must be closed. □

(*b*) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1.

Proof. C_1 and C_2 are each sets of isolated points and so are closed. C_1 and C_2 are both countable sets, so $C_1 + C_2$ must be countable also.

We will show that $C_1 + C_2$ is dense in $[0, 1]$. Let $[x]$ denote the greatest integer less than or equal to x and let $(x) = x - [x]$ denote the fractional part of x. Let N be any positive integer and consider the $N + 1$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_{N+1}$ defined by

$$\alpha_i = (i\alpha) = i\alpha - [i\alpha], \quad 1 \leq i \leq N + 1.$$

Note that each α_i belongs to $C_1 + C_2$, since $i\alpha \in C_2$ and $-[i\alpha] \in C_1$. Note also that each α_i is distinct, since $\alpha_i = \alpha_j$ for $i \neq j$ implies that

$$i\alpha - [i\alpha] = j\alpha - [j\alpha]$$

so

$$\alpha = \frac{[i\alpha] - [j\alpha]}{i - j},$$

and this would imply that α is rational, which is a contradiction.

Now consider the N segments

$$I_k = \left(\frac{k-1}{N}, \frac{k}{N} \right), \quad k = 1, 2, \ldots, N.$$

Each α_i must belong to one of these segments, since each α_i cannot equal one of the endpoints (since that would imply that α is rational) and all other points in $[0, 1]$ belong to some I_k.

Since there are $N + 1$ points and only N intervals, there must be at least one I_k which contains at least two points α_i and α_j with $i \neq j$. Let α_j be the larger of the two. Then $\alpha_j - \alpha_i < 1/N$. But $\alpha_j - \alpha_i \in C_1 + C_2$, so the segment $(0, 1/N)$ contains a point β in $C_1 + C_2$ for each positive integer N. Then the segment $((k-1)/N, k/N)$ also contains such a point for each N, since $(k-1)\beta$ is in this interval and also in $C_1 + C_2$.

Now every segment $(a, b) \in [0, 1]$ must contain a segment of the form $((k-1)/N, k/N)$ for some positive integers N and k. And this segment has been shown to contain at least one element of $C_1 + C_2$. Therefore $C_1 + C_2$ is dense in $[0, 1]$. Finally, since every point in R^1 can be written as the sum of a number in $[0, 1]$ and an integer, it follows that $C_1 + C_2$ is dense in all of R^1.

Finally, $C_1 + C_2$ cannot be closed since it is a dense proper subset of R^1. □

4.26 Exercise 26

Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Prove also that f is continuous if h is continuous.

Show that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Proof. Let h be uniformly continuous. Since g is a continuous one-to-one mapping defined on the compact metric space Y, we know that the inverse mapping $g^{-1}: g(Y) \to Y$ is continuous by Theorem 4.17. Note also that for all x in X, $f(x) = g^{-1}(h(x))$.

Now, since g is continuous on a compact metric space, its image $g(Y)$ is also compact by Theorem 4.14. But then g^{-1} has as its domain the compact metric space $g(Y)$, so g^{-1} is also uniformly continuous by Theorem 4.19. Therefore f is the composition of two uniformly continuous functions, and so is uniformly continuous by Exercise 4.12.

Next, if h is continuous (but not necessarily uniformly continuous), then f must be continuous by the same argument that was given above.

Finally, the compactness of Y is necessary for these results to hold in general. To see this, let $X = Z = [0, 1]$ and $Y = \{0\} \cup [1, \infty)$. Define $f: X \to Y$ by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

and $g: Y \to Z$ by

$$g(x) = \begin{cases} \frac{1}{x}, & \text{if } x \geq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

Now X and Z are compact, g is a continuous one-to-one mapping, and h is the identity function on $[0, 1]$ and so is continuous, but f is clearly not continuous. □