Selected Solutions to Hoffman and Kunze's Linear Algebra Second Edition

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This document lives at:

https://www.gregkikola.com/projects/guides/

You can find the LaTeX source code on GitHub at:

https://github.com/gkikola/sol-hoffman-kunze

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Preface

This is an unofficial solution guide to the book *Linear Algebra*, Second Edition, by Kenneth Hoffman and Ray Kunze. It is intended for students who are studying linear algebra using Hoffman and Kunze's text. I encourage students who use this guide to first attempt each exercise on their own before looking up the solution, as doing exercises is an essential part of learning mathematics.

In writing this guide, I have avoided using techniques or results before the point at which they are introduced in the text. My solutions should therefore be accessible to someone who is reading through Hoffman and Kunze for the first time.

Given the large number of exercises, errors are unavoidable in a work such as this. I have done my best to proofread each solution, but mistakes will get through nonetheless. If you find one, please feel free to tell me about it via email: gkikola@gmail.com. I appreciate any corrections or feedback.

Please know that this guide is currently unfinished. I am slowly working on adding the remaining chapters, but this will be done at my own pace. If you need a solution to an exercise that I have not included, try typing the problem statement into a web search engine such as Google; it is likely that someone else has already posted a solution.

I am deeply grateful to the authors, Kenneth Hoffman and Ray Kunze, for producing a well-organized and comprehensive book that is a pleasure to read.

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Chapter 1

Linear Equations

1.2 Systems of Linear Equations

1.2.1 Exercise 1

Verify that the set of complex numbers described in Example 4 is a subfield of C.

Solution. The set in Example 4 consisted of all complex numbers of the form $x + y\sqrt{2}$, where x and y are rational. Call this set F.

Note that $0 = 0 + 0\sqrt{2} \in F$ and $1 = 1 + 0\sqrt{2} \in F$. If $\alpha = a + b\sqrt{2}$ and $\beta = c + d\sqrt{2}$ are any elements of F, then

$$\alpha + \beta = (a + c) + (b + d)\sqrt{2} \in F,$$

and

$$-\alpha = -a - b\sqrt{2} \in F.$$

We also have

$$\begin{aligned} \alpha\beta &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in F \end{aligned}$$

and, provided α is nonzero,

$$\alpha^{-1} = \frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in F.$$

Since F contains 0 and 1 and is closed under addition, multiplication, additive inverses, and multiplicative inverses, F is a subfield of C. □

1.2.2 Exercise 2

Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \qquad & 3x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \qquad & x_1 + x_2 &= 0 \end{aligned}$$

Solution. The systems are equivalent. For the first system, we can write

$$\begin{aligned} x_1 - x_2 &= (3x_1 + x_2) - 2(x_1 + x_2) = 0, \\ 2x_1 + x_2 &= \tfrac{1}{2}(3x_1 + x_2) + \tfrac{1}{2}(x_1 + x_2) = 0. \end{aligned}$$

And for the second system,

$$\begin{aligned} 3x_1 + x_2 &= \tfrac{1}{3}(x_1 - x_2) + \tfrac{4}{3}(2x_1 + x_2) = 0, \\ x_1 + x_2 &= -\tfrac{1}{3}(x_1 - x_2) + \tfrac{2}{3}(2x_1 + x_2) = 0. \end{aligned}$$ □

1.2.3 Exercise 3

Test the following systems of equations as in Exercise 1.2.2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

Solution. For the first system, we have

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= -(x_1 - x_3) + (x_2 + 3x_3) = 0, \\ x_1 + 3x_2 + 8x_3 &= (x_1 - x_3) + 3(x_2 + 3x_3) = 0, \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= \tfrac{1}{2}(x_1 - x_3) + (x_2 + 3x_3) = 0. \end{aligned}$$

For the second system, we have

$$\begin{aligned} x_1 - x_3 &= -\tfrac{3}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3), \\ x_2 + 3x_3 &= \tfrac{1}{4}(-x_1 + x_2 + 4x_3) + \tfrac{1}{4}(x_1 + 3x_2 + 8x_3) + 0(\tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3). \end{aligned}$$

So, the two systems are equivalent. □

1.2.4 Exercise 4

Test the following systems as in Exercise 1.2.2.

$$\begin{aligned} 2x_1 + (-1 + i)x_2 \qquad + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad \begin{aligned} \left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

Solution. Call the equations in the system on the left L_1 and L_2, and the equations on the right R_1 and R_2. If $R_1 = aL_1 + bL_2$ then, by equating the coefficients of x_3, we get

$$-i = -2ib,$$

which implies that $b = 1/2$. By equating the coefficients of x_1, we get

$$1 + \frac{i}{2} = 2a,$$

so that

$$a = \frac{1}{2} + \frac{1}{4}i.$$

Now, comparing the coefficients of x_4, we find that

$$-1 = a + 5b = \frac{1}{2} + \frac{1}{4}i + \frac{5}{2} = 3 + \frac{1}{4}i,$$

which is clearly a contradiction. Therefore the two systems are *not* equivalent. □

1.2.5 Exercise 5

Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Verify that the set F, together with these two operations, is a field.

Solution. From the symmetry in the tables, we see that both operations are commutative.

By considering all eight possibilities, one can see that $(a+b)+c = a+(b+c)$. And one may in a similar way verify that $(ab)c = a(bc)$, so that associativity holds for the two operations.

$0 + 0 = 0$ and $0 + 1 = 1$ so F has an additive identity. Similarly, $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$ so F has a multiplicative identity.

The additive inverse of 0 is 0 and the additive inverse of 1 is 1. The multiplicative inverse of 1 is 1. So F has inverses.

Lastly, by considering the eight cases, one may verify that $a(b+c) = ab+ac$. Therefore distributivity of multiplication over addition holds and F is a field. □

1.2.7 Exercise 7

Prove that each subfield of the field of complex numbers contains every rational number.

Proof. Let F be a subfield of C and let $r = m/n$ be any rational number, written in lowest terms. F must contain 0 and 1, so if $r = 0$ then we are done. Now assume r is nonzero.

Since $1 \in F$, and F is closed under addition, we know that $1 + 1 = 2 \in F$. And, if the integer k is in F, then $k + 1$ is also in F. By induction, we see that all positive integers belong to F. We also know that all negative integers are in F because F is closed under additive inverses. So, in particular, $m \in F$ and $n \in F$.

Now F is closed under multiplicative inverses, so $n \in F$ implies $1/n \in F$. Finally, closure under multiplication shows that $m \cdot (1/n) = m/n = r \in F$. Since r was arbitrary, we can conclude that all rational numbers are in F. □

1.2.8 Exercise 8

Prove that each field of characteristic zero contains a copy of the rational number field.

Proof. Let F be a field of characteristic zero. Define the map $f\colon Q \to F$ (where Q denotes the rational numbers) as follows. Let $f(0) = 0_F$ and $f(1) = 1_F$, where 0_F and 1_F are the additive and multiplicative identities, respectively, of F. Given a positive integer n, define $f(n) = f(n-1) + 1_F$ and $f(-n) = -f(n)$. If a rational number $r = m/n$ is not an integer, define $f(r) = f(m) \cdot (f(n))^{-1}$.

First we show that the function f preserves addition and multiplication. A simple induction argument will show that, in the case of integers m and n, we have

$$f(m+n) = f(m) + f(n) \quad \text{and} \quad f(mn) = f(m)f(n).$$

Now let $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ be rational numbers in lowest terms. Then, by the definition of f,

$$\begin{aligned} f(r_1 + r_2) &= f((m_1n_2 + m_2n_1)/(n_1n_2)) \\ &= f(m_1n_2 + m_2n_1)f(n_1n_2)^{-1} \\ &= (f(m_1)f(n_2) + f(m_2)f(n_1))f(n_1)^{-1}f(n_2)^{-1} \\ &= f(m_1)f(n_1)^{-1} + f(m_2)f(n_2)^{-1} \\ &= f(r_1) + f(r_2). \end{aligned}$$

Likewise,

$$f(r_1r_2) = f(m_1)f(m_2)f(n_1)^{-1}f(n_2)^{-1} = f(r_1)f(r_2).$$

(Formally, this shows that f is a ring homomorphism.)

We will next show that the function f is one-to-one. If $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ are rational numbers in lowest terms, then $f(r_1) = f(r_2)$ implies

$$f(m_1)f(n_1)^{-1} = f(m_2)f(n_2)^{-1}$$

or

$$f(m_1)f(n_2) = f(m_2)f(n_1).$$

This implies

$$f(m_1n_2) = f(m_2n_1).$$

Now if $m_1n_2 \neq m_2n_1$, then this would imply that F does not have characteristic zero. So $m_1n_2 = m_2n_1$ and so $r_1 = r_2$.

What we have shown is that every rational number corresponds to a distinct element of F, and that the operations of addition and multiplication of rational numbers is preserved by this correspondence. So F contains a copy of Q. $\square$

1.3 Matrices and Elementary Row Operations

1.3.1 Exercise 1

Find all solutions to the system of equations

$$\begin{aligned}(1-i)x_1 - ix_2 &= 0\\ 2x_1 + (1-i)x_2 &= 0.\end{aligned} \tag{1.1}$$

Solution. Using elementary row operations, we get

$$\begin{bmatrix}1-i & -i\\ 2 & 1-i\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1-i & -i\\ 1 & \frac{1}{2}-\frac{1}{2}i\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}0 & 0\\ 1 & \frac{1}{2}-\frac{1}{2}i\end{bmatrix}.$$

So the system in (1.1) is equivalent to

$$x_1 + \left(\frac{1}{2}-\frac{1}{2}i\right)x_2 = 0.$$

Therefore, if c is any complex scalar, then $x_1 = (-1+i)c$ and $x_2 = 2c$ is a solution to (1.1). □

1.3.2 Exercise 2

If

$$A = \begin{bmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix}$$

find all solutions of $AX = 0$ by row-reducing A.

Solution. We get

$$\begin{bmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 2 & 1 & 1\\ 1 & -3 & 0\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 0 & \frac{5}{3} & -\frac{1}{3}\\ 0 & -\frac{8}{3} & -\frac{2}{3}\end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix}1 & -\frac{1}{3} & \frac{2}{3}\\ 0 & 1 & -\frac{1}{5}\\ 0 & -\frac{8}{3} & -\frac{2}{3}\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}1 & 0 & \frac{3}{5}\\ 0 & 1 & -\frac{1}{5}\\ 0 & 0 & -\frac{6}{5}\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}1 & 0 & \frac{3}{5}\\ 0 & 1 & -\frac{1}{5}\\ 0 & 0 & 1\end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Thus $AX = 0$ has only the trivial solution. □

1.3.3 Exercise 3

If

$$A = \begin{bmatrix}6 & -4 & 0\\ 4 & -2 & 0\\ -1 & 0 & 3\end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X.)

Solution. The matrix equation $AX = 2X$ corresponds to the system of linear equations

$$\begin{aligned} 6x_1 - 4x_2 \quad\quad &= 2x_1 \\ 4x_1 - 2x_2 \quad\quad &= 2x_2 \\ -1x_1 \quad\quad + 3x_3 &= 2x_3, \end{aligned}$$

or, equivalently,

$$\begin{aligned} 4x_1 - 4x_2 \quad\quad &= 0 \\ 4x_1 - 4x_2 \quad\quad &= 0 \\ -1x_1 \quad\quad + x_3 &= 0. \end{aligned}$$

This system is homogeneous, and can be represented by the equation $BX = 0$, where B is given by

$$B = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

B can be row-reduced:

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore any solution of $AX = 2X$ will have the form

$$(x_1, x_2, x_3) = (a, a, a) = a(1, 1, 1),$$

where a is a scalar.

Similarly, the equation $AX = 3X$ can be solved by row-reducing

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, solutions of $AX = 3X$ have the form

$$(x_1, x_2, x_3) = (0, 0, b) = b(0, 0, 1),$$

where b is a scalar. □

1.3.4 Exercise 4

Find a row-reduced matrix which is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

Solution. Using the elementary row operations, we get

$$\begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & -1+i & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 1+i & -1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is row-equivalent to A. □

1.3.5 Exercise 5

Prove that the following two matrices are *not* row-equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$

Proof. By performing row operations on the first matrix, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that this matrix is row-equivalent to the identity matrix. The corresponding system of equations has only the trivial solution.

For the second matrix, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The system of equations corresponding to this matrix has nontrivial solutions. Therefore the two matrices are not row-equivalent. □

1.3.6 Exercise 6

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix with complex entries. Suppose that A is row-reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

Proof. One possibility is the zero matrix,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If A is not the zero matrix, then it has at least one nonzero row. If it has exactly one nonzero row, then in order to satisfy the given constraints, the nonzero row will have a 1 in the first column and a -1 in the second. This gives two possibilities,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Finally, if A has two nonzero rows, then it must be the identity matrix or the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, but neither of these are valid since the sum of the entries is nonzero in each case. Thus there are only the three possibilities given above. □

1.3.7 Exercise 7

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Proof. We can, without loss of generality, assume that the matrix has only two rows, since any additional rows could just be ignored in the procedure that follows. Let this matrix be given by

$$A_0 = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

First, add -1 times row 2 to row 1 to get the matrix

$$A_1 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}.$$

Next, add row 1 to row 2 to get

$$A_2 = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 & \cdots & a_n - b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

and then add -1 times row 2 to row 1, which gives

$$A_3 = \begin{bmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

For the final step, multiply row 1 by -1 to get

$$A_4 = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

We can see that A_4 has the same entries as A_0 but with the rows interchanged. And only a finite number of elementary row operations of the first two kinds were performed. □

1.3.8 Exercise 8

Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix over the field F. Prove the following.

(a) If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.

Proof. This is clear, since the equation $0x_1 + 0x_2 = 0$ is satisfied for any $(x_1, x_2) \in F^2$ (note that in any field, $0x = (1-1)x = x - x = 0$). $\square$

(b) If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.

Proof. First suppose $bd \neq 0$. Then we can perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad - bc & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ bc & bd \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, A is row-equivalent to the 2×2 identity matrix.

On the other hand, if $bd = 0$ then one of b or d is zero (but not both). If $b = 0$, then $ad \neq 0$ and we get

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $d = 0$, then $bc \neq 0$ and we have

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & 0 \\ a & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that, in every case, A is row-equivalent to the identity matrix. Therefore $AX = 0$ has only the trivial solution. $\square$

(c) If $ad - bc = 0$ and some entry of A is different from 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$, $x_2 = yx_2^0$.

Proof. Since one of the entries a, b, c, d is nonzero, we can assume without loss of generality that a is nonzero (because, if the first row is zero then we could simply interchange the rows and relabel the entries; and if the only nonzero entry occurs in the second column, then we could interchange the columns which would correspond to relabeling x_1 and x_2).

Keeping in mind that a is nonzero, we perform the following row-reduction.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

Since $ad - bc = 0$, the second row of this final matrix is zero, and we see that there are nontrivial solutions. If we let

$$x_1^0 = b \quad \text{and} \quad x_2^0 = -a,$$

then (x_1, x_2) is a solution if and only if $x_1 = yx_1^0$ and $x_2 = yx_2^0$ for some $y \in F$. $\square$

1.4 Row-Reduced Echelon Matrices

1.4.1 Exercise 1

Find all solutions to the following system of equations by row-reducing the coefficient matrix:

$$\begin{aligned} \tfrac{1}{3}x_1 + 2x_2 - 6x_3 &= 0 \\ -4x_1 + 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -\tfrac{7}{3}x_1 + 2x_2 - \tfrac{8}{3}x_3 &= 0 \end{aligned}$$

Solution. The coefficient matrix reduces as follows:

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -\frac{67}{24} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 16 & -\frac{134}{3} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $x_3 = 24t$, we see that all solutions have the form

$$x_1 = 30t, \quad x_2 = 67t, \quad \text{and} \quad x_3 = 24t,$$

where t is an arbitrary scalar. □

1.4.2 Exercise 2

Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix}.$$

What are the solutions of $AX = 0$?

Solution. Performing row-reduction on A gives

$$\begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -i \\ 0 & 2+2i \\ 0 & i \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & i \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and this last matrix is in row-reduced echelon form. Therefore the homogeneous system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$. □

1.4.3 Exercise 3

Describe explicitly all 2×2 row-reduced echelon matrices.

Solution. If a 2×2 matrix has no nonzero rows, then it is the zero matrix,

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is in row-reduced echelon form.

Next, if a 2×2 matrix has exactly one nonzero row, then in order to be in row-reduced echelon form, the nonzero row must be in row 1 and it must start with an entry of 1. There are two possibilities,

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where a is an arbitrary scalar.

Lastly, if a 2×2 matrix in row-reduced echelon form has two nonzero rows, then the diagonal entries must be 1 and the other entries 0, so we get the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These are the only possibilities. □

1.4.4 Exercise 4

Consider the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 \qquad + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2. \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

Solution. We perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & -2 & 2 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this we see that the original system of equations has solutions. All solutions are of the form

$$x_1 = -t + \frac{1}{2}, \quad x_2 = t - \frac{1}{2}, \quad \text{and} \quad x_3 = t,$$

for some scalar t. □

1.4.5 Exercise 5

Give an example of a system of two linear equations in two unknowns which has no solution.

Solution. We can find such a system by ensuring that the coefficients in one equation are a multiple of the other, while the constant term is not the same multiple. For example, one such system is

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -3x_1 - 6x_2 &= 5. \end{aligned}$$

This system has no solutions since the augmented matrix is row-equivalent to a matrix in which one row consists of zero entries everywhere but the rightmost column. □

1.4.6 Exercise 6

Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

Solution. Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 9 & -6 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since the first nonzero entry in the bottom row of the last matrix is in the rightmost column, the corresponding system of equations has no solution. Therefore the original system of equations also has no solution. □

1.4.7 Exercise 7

Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

Solution. The augmented matrix can be row-reduced as follows:

$$\begin{bmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns with a leading 1 correspond to the variables x_1, x_2, and x_5, so these variables will depend on the remaining two variables, which can take any value. Therefore all solutions have the form

$$x_1 = 2s - t + 1, \quad x_2 = t - s + 2, \quad x_3 = s, \quad x_4 = t, \quad \text{and} \quad x_5 = 1,$$

where s and t are arbitrary scalars. □

1.4.8 Exercise 8

Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

Solution. We will perform row-reduction on the augmented matrix:

$$\begin{bmatrix} 3 & -1 & 2 & y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 2 & 1 & 1 & y_2 \\ 1 & -3 & 0 & y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & \frac{5}{3} & -\frac{1}{3} & -\frac{2}{3}y_1 + y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}y_1 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{1}{3}y_1 + y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & -\frac{6}{5} & -\frac{7}{5}y_1 + \frac{8}{5}y_2 + y_3 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}y_1 + \frac{1}{5}y_2 \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5}y_1 + \frac{3}{5}y_2 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2}y_1 + y_2 + \frac{1}{2}y_3 \\ 0 & 1 & 0 & -\frac{1}{6}y_1 + \frac{1}{3}y_2 - \frac{1}{6}y_3 \\ 0 & 0 & 1 & \frac{7}{6}y_1 - \frac{4}{3}y_2 - \frac{5}{6}y_3 \end{bmatrix}.$$

Since every row contains a nonzero entry in the first three columns, the system of equations $AX = Y$ is consistent regardless of the values of y_1, y_2, and y_3. Therefore $AX = Y$ has a unique solution for any triple (y_1, y_2, y_3). □

1.4.9 Exercise 9

Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

Solution. Row-reduction on the augmented matrix gives

$$\begin{bmatrix} 3 & -6 & 2 & -1 & y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} & \frac{2}{3}y_1 + y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3}y_1 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}y_1 + y_4 \end{bmatrix} \xrightarrow{(2)}$$
$$\begin{bmatrix} 1 & -2 & 0 & -1 & \frac{1}{7}y_1 - \frac{2}{7}y_2 \\ 0 & 0 & 1 & 1 & \frac{2}{7}y_1 + \frac{3}{7}y_2 \\ 0 & 0 & 0 & 0 & -\frac{2}{7}y_1 - \frac{3}{7}y_2 + y_3 \\ 0 & 0 & 0 & 0 & -\frac{3}{7}y_1 - \frac{1}{7}y_2 + y_4 \end{bmatrix}.$$

So, in order for the system $AX = Y$ to have a solution, we need (y_1, y_2, y_3, y_4) to satisfy

$$\begin{aligned} -\tfrac{2}{7}y_1 - \tfrac{3}{7}y_2 + y_3 \quad &= 0 \\ -\tfrac{3}{7}y_1 - \tfrac{1}{7}y_2 \quad + y_4 &= 0. \end{aligned}$$

To determine the conditions on Y, we row-reduce the coefficient matrix for this system.

$$\begin{bmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ -\frac{3}{7} & -\frac{1}{7} & 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{7}{2} & 0 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}.$$

From this we see that in order for $AX = Y$ to have a solution, (y_1, y_2, y_3, y_4) must take the form

$$(y_1, y_2, y_3, y_4) = (3t - s, 3s - 2t, s, t),$$

where s and t are arbitrary. □

1.5 Matrix Multiplication

1.5.1 Exercise 1

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Compute ABC and CAB.

Solution. We get

$$\begin{aligned} ABC &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} CAB &= \begin{bmatrix} 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \end{aligned}$$

□

1.5.2 Exercise 2

Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}.$$

Verify directly that $A(AB) = A^2B$.

Solution. We have

$$\begin{aligned} A(AB) &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned}
A^2B &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}.
\end{aligned}$$

So $A(AB) = A^2B$ as expected. □

1.5.3 Exercise 3

Find two different 2×2 matrices A such that $A^2 = 0$ but $A \neq 0$.

Solution. Two possibilities are

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Both of these are nonzero matrices that satisfy $A^2 = 0$. □

1.5.4 Exercise 4

For the matrix A of Exercise 1.5.2, find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I.$$

Solution. We want to reduce

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

to the identity matrix. To start, we can use two elementary row operations of the second kind to get 0 in the bottom two entries of column 1. Performing the same operations on the identity matrix gives

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Then

$$E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}.$$

Next, we can use a row operation of the first kind to make the central entry into a 1:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so that} \quad E_3E_2E_1A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -2 \end{bmatrix}.$$

Continuing in this way, we get

$$E_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix},$$

so that

$$E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Then

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{so that} \quad E_6E_5E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_8 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$E_8E_7E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus each of $E_1, E_2, \ldots, E_8$ are elementary matrices, and they are such that $E_8 \cdots E_2E_1A = I$. □

1.5.5 Exercise 5

Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

Is there a matrix C such that $CA = B$?

Solution. Suppose there is, and let

$$C = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}.$$

Then

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

This leads to the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 3, & c_4 + 2c_5 + c_6 &= -4, \\ -c_1 + 2c_2 \phantom{{}+c_3} &= 1, & -c_4 + 2c_5 \phantom{{}+c_6} &= 4. \end{aligned}$$

This system has solutions

$$(c_1, c_2, c_3, c_4, c_5, c_6) = (1 - 2s, 1 - s, 4s, 2t - 4, t, -4t).$$

For example, taking $s = t = -1$, we get the matrix

$$C = \begin{bmatrix} 3 & 2 & -4 \\ -6 & -1 & 4 \end{bmatrix},$$

and one can easily verify that $CA = B$. □

1.5.6 Exercise 6

Let A be an $m \times n$ matrix and B an $n \times k$ matrix. Show that the columns of $C = AB$ are linear combinations of the columns of A. If $\alpha_1, \ldots, \alpha_n$ are the columns of A and $\gamma_1, \ldots, \gamma_k$ are the columns of C, then

$$\gamma_j = \sum_{r=1}^{n} B_{rj}\alpha_r.$$

Proof. Let A, B, C be as stated. By the definition of matrix multiplication, we have

$$\begin{aligned} \gamma_j &= \begin{bmatrix} A_{11}B_{1j} + A_{12}B_{2j} + \cdots + A_{1n}B_{nj} \\ A_{21}B_{1j} + A_{22}B_{2j} + \cdots + A_{2n}B_{nj} \\ \vdots \\ A_{m1}B_{1j} + A_{m2}B_{2j} + \cdots + A_{mn}B_{nj} \end{bmatrix} \\ &= B_{1j}\begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} + B_{2j}\begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} + \cdots + B_{nj}\begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} \\ &= B_{1j}\alpha_1 + B_{2j}\alpha_2 + \cdots + B_{nj}\alpha_n = \sum_{r=1}^{n} B_{rj}\alpha_r. \end{aligned}$$

Therefore the columns of $C = AB$ are linear combinations of the columns of A. □

1.6 Invertible Matrices

1.6.1 Exercise 1

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

Solution. We can perform elementary row operations on A, while performing the same operations on I, in order to find R and P:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & -4 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 8 & 11 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

Therefore,

$$R = \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix},$$

and $R = PA$. □

1.6.2 Exercise 2

Do Exercise 1.6.1, but with

$$A = \begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}.$$

Solution. We proceed as before:

$$\begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 2 & 0 & i \\ i & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 6 & 3i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -i & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 1+3i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & -i & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & \frac{3}{2}-\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{6}-\frac{1}{2}i & \frac{1}{3} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}i \\ 0 & 1 & \frac{1}{2}i \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{30}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ 0 & -\frac{3}{10}-\frac{1}{10}i & \frac{1}{10}-\frac{3}{10}i \\ -\frac{1}{3}i & \frac{1}{5}+\frac{1}{15}i & \frac{3}{5}+\frac{1}{5}i \end{bmatrix}.$$

So

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, \quad P = \frac{1}{30}\begin{bmatrix} 10 & 1-3i & 3-9i \\ 0 & -9-3i & 3-9i \\ -10i & 6+2i & 18+6i \end{bmatrix},$$

and $R = PA$. $\square$

1.6.3 Exercise 3

For each of the two matrices

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

Solution. For the first matrix, row-reduction gives

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

and we see that the original matrix is not invertible since it is row-equivalent to a matrix having a row of zeros.

For the second matrix, we get

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 4\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & -2\\ 0 & 1 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -2\\ 0 & 5 & -2\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -3 & 1 & 0\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 8\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -3 & 1 & -5\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 1\\ -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{8} & \frac{1}{8} & -\frac{5}{8}\end{bmatrix}.$$

From this we see that the original matrix is invertible and its inverse is the matrix

$$\frac{1}{8}\begin{bmatrix}8 & 0 & 8\\ -6 & 2 & -2\\ -3 & 1 & -5\end{bmatrix}. \qquad \square$$

1.6.4 Exercise 4

Let

$$A = \begin{bmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{bmatrix}.$$

For which X does there exist a scalar c such that $AX = cX$?

Solution. Let

$$X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Then $AX = cX$ implies

$$\begin{aligned} 5x_1 &= cx_1\\ x_1 + 5x_2 &= cx_2\\ x_2 + 5x_3 &= cx_3, \end{aligned}$$

and this is a homogeneous system of equations with coefficient matrix

$$B = \begin{bmatrix} 5-c & 0 & 0 \\ 1 & 5-c & 0 \\ 0 & 1 & 5-c \end{bmatrix}.$$

If $c = 5$ then $(x_1, x_2, x_3) = (0, 0, t)$ for some scalar t, so this gives one possibility for X. If we assume $c \neq 5$, then the matrix B can be row-reduced to the identity matrix, so that $X = 0$ is then the only possibility. Therefore, there is a scalar c with $AX = cX$ if and only if

$$X = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix},$$

for some arbitrary scalar t. □

1.6.5 Exercise 5

Discover whether

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible, and find A^{-1} if it exists.

Solution. We proceed in the usual way:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Thus A is invertible and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$ □

1.6.6 Exercise 6

Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible.

Proof. Let

$$A = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \end{bmatrix}$$

so that

$$C = AB = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}.$$

Suppose C has an inverse. Then C is row-equivalent to the identity matrix, and so cannot be row-equivalent to a matrix having a row of zeros. Consequently, each of a, b, c, and d must be nonzero, since otherwise C would be row-equivalent to such a matrix.

But, since a and b are nonzero, we can multiply the second row of C by a/b to get the row-equivalent matrix

$$\begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ac & ad \\ ac & ad \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ac & bd \\ 0 & 0 \end{bmatrix},$$

which is clearly not invertible. Therefore C cannot have an inverse. □

1.6.7 Exercise 7

Let A be an $n \times n$ (square) matrix. Prove the following two statements:

(a) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.

Proof. Since $AB = 0$ and A is invertible, we can multiply on the left by A^{-1} to get

$$B = A^{-1}0.$$

But the product on the right is clearly the $n \times n$ zero matrix, so $B = 0$. □

(b) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

Proof. If A is not invertible, then the homogeneous system of equations $AX = 0$ has a nontrivial solution X_0. Let B be the matrix whose first column is X_0 and whose other entries are all zero, and consider the product AB.

The entries in the first column of AB must be zero since the first column is just AX_0, and the remaining entries must be zero since all other columns are the product of A with a zero column. Thus the proof is complete. □

1.6.8 Exercise 8

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove, using elementary row operations, that A is invertible if and only if

$$ad - bc \neq 0.$$

Proof. First, if A is invertible, then one of a, b, c, or d must be nonzero. If $a \neq 0$, then we can reduce

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix},$$

and we must have $ad - bc \neq 0$ since otherwise A could not be row-equivalent to the identity matrix, contradicting Theorem 12.

If, instead, $a = 0$ then we must have $b \neq 0$ since otherwise A would have a row of zeros and could not be row-equivalent to the identity matrix. So we can proceed:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix},$$

and we see that we must have $c \neq 0$. Thus $ad - bc = -bc \neq 0$. This completes the first half of the proof.

Conversely, assume that $ad - bc \neq 0$. If $d \neq 0$ then A can be reduced to get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} ad & bd \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} ad - bc & 0 \\ c & d \end{bmatrix} \xrightarrow{(1)}$$
$$\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and A is row-equivalent to the identity matrix. On the other hand, if $d = 0$ then b and c must be nonzero and we get

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that A is again row-equivalent to the identity matrix. In either case, A must be invertible by Theorem 12. $\square$

1.6.9 Exercise 9

An $n \times n$ matrix A is called **upper-triangular** if $A_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

Proof. Let A be an $n \times n$ upper-triangular matrix.

First, suppose every entry on the main diagonal of A is nonzero, and consider the homogeneous linear system $AX = 0$:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\ A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\ &\vdots \\ A_{nn}x_n &= 0. \end{aligned}$$

Since A_{nn} is nonzero, the last equation implies that $x_n = 0$. Then, since $A_{n-1,n-1}$ is nonzero, the second-to-last equation implies that $x_{n-1} = 0$. Continuing in this way, we see that $x_i = 0$ for each $i = 1, 2, \ldots, n$. Therefore the system $AX = 0$ has only the trivial solution, hence A is invertible.

Conversely, suppose A is invertible. Then A cannot contain any zero rows, nor can A be row-equivalent to a matrix with a row of zeros. This implies that $A_{nn} \neq 0$. Consider $A_{n-1,n-1}$. If $A_{n-1,n-1}$ is zero, then by dividing row n by A_{nn}, and then by adding $-A_{n-1,n}$ times row n to row $n-1$, we see that A is row-equivalent to a matrix whose $(n-1)$st row is all zeros. This is a contradiction, so $A_{n-1,n-1} \neq 0$. In the same manner, we can show that $A_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. Thus all entries on the main diagonal of A are nonzero. $\square$

1.6.11 Exercise 11

Let A be an $m \times n$ matrix. Show that by means of a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both 'row-reduced echelon' and 'column-reduced echelon,' i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$ if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Proof. By Theorem 5, A is row-equivalent to a row-reduced echelon matrix R_0. And, by the second corollary to Theorem 12, there is an invertible $m \times m$ matrix P such that $R_0 = PA$.

Results that are analogous to Theorems 5 and 12 (with similar proofs) hold for column-reduced echelon matrices, so there is a matrix R which is column-equivalent to R_0 and an invertible $n \times n$ matrix Q such that $R = R_0Q$. Then $R = PAQ$ and we see that, through a finite number of elementary row and/or column operations, A passes to a matrix R that is both row- and column-reduced echelon. $\square$

Chapter 2

Vector Spaces

2.1 Vector Spaces

2.1.1 Exercise 1

If F is a field, verify that F^n (as defined in Example 1) is a vector space over the field F.

Proof. We need to check that addition and scalar multiplication, as defined in Example 1, satisfy conditions (3) and (4) of the definition. Let

$$\begin{aligned}\alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n),\\ \beta &= (\beta_1, \beta_2, \ldots, \beta_n),\end{aligned}$$

and

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$$

be arbitrary vectors in F^n. From the commutativity of addition in F, we have

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) = (\beta_1 + \alpha_1, \ldots, \beta_n + \alpha_n) = \beta + \alpha,$$

so addition is commutative in F^n. Similarly, by associativity of addition in F, we have

$$\begin{aligned}\alpha + (\beta + \gamma) &= (\alpha_1 + (\beta_1 + \gamma_1), \ldots, \alpha_n + (\beta_n + \gamma_n))\\ &= ((\alpha_1 + \beta_1) + \gamma_1, \ldots, (\alpha_n + \beta_n) + \gamma_n)\\ &= (\alpha + \beta) + \gamma,\end{aligned}$$

and associativity holds in F^n. The unique 0 vector is

$$0 = (0, 0, \ldots, 0),$$

and it is clear that $\alpha + 0 = \alpha$. The unique additive inverse of α is given by

$$-\alpha = (-\alpha_1, -\alpha_2, \ldots, -\alpha_n),$$

and certainly $\alpha + (-\alpha) = 0$. The conditions in (3) are satisfied.

Now let c and d be scalars in F. Then

$$\begin{aligned}1\alpha &= (1\alpha_1, \dots, 1\alpha_n) = (\alpha_1, \dots, \alpha_n) = \alpha, \\ (cd)\alpha &= ((cd)\alpha_1, \dots, (cd)\alpha_n) = (c(d\alpha_1), \dots, c(d\alpha_n)) = c(d\alpha), \\ c(\alpha + \beta) &= (c(\alpha_1 + \beta_1), \dots, c(\alpha_n + \beta_n)) \\ &= (c\alpha_1 + c\beta_1, \dots, c\alpha_n + c\beta_n) \\ &= c\alpha + c\beta,\end{aligned}$$

and

$$\begin{aligned}(c + d)\alpha &= ((c + d)\alpha_1, \dots, (c + d)\alpha_n) \\ &= (c\alpha_1 + d\alpha_1, \dots, c\alpha_n + d\alpha_n) \\ &= c\alpha + d\alpha,\end{aligned}$$

so the conditions in (4) are satisfied. Therefore F^n is a vector space over F. □

2.1.2 Exercise 2

If V is a vector space over the field F, verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors α_1, α_2, α_3, and α_4 in V.

Proof. We only need to make use of commutativity and associativity of vector addition:

$$\begin{aligned}(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) &= (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \\ &= \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \\ &= \alpha_2 + [(\alpha_1 + \alpha_3) + \alpha_4] \\ &= [\alpha_2 + (\alpha_1 + \alpha_3)] + \alpha_4 \\ &= [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4.\end{aligned}$$ □

2.1.3 Exercise 3

If C is the field of complex numbers, which vectors in C^3 are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$?

Solution. A vector $\alpha = (y_1, y_2, y_3)$ is a linear combination of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$ if there are scalars x_1, x_2, x_3 such that

$$x_1(1, 0, -1) + x_2(0, 1, 1) + x_3(1, 1, 1) = \alpha,$$

which leads to the following system of equations:

$$\begin{aligned}x_1 \qquad + x_3 &= y_1 \\ x_2 + x_3 &= y_2 \\ -x_1 + x_2 + x_3 &= y_3.\end{aligned}$$

Since the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

is row-equivalent to the identity matrix, this system of equations has a solution for each α. Therefore, all vectors in C^3 are linear combinations of the vectors $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$. □

2.1.4 Exercise 4

Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$\begin{aligned} (x, y) + (x_1, y_1) &= (x + x_1, y + y_1) \\ c(x, y) &= (cx, y). \end{aligned}$$

Is V, with these operations, a vector space over the field of real numbers?

Solution. No, V is not a vector space. Most of the conditions are satisfied, but distributivity over scalar addition fails when y is nonzero:

$$(c + d)(x, y) = ((c + d)x, y) = (cx + dx, y)$$

but

$$c(x, y) + d(x, y) = (cx, y) + (dx, y) = (cx + dx, 2y).$$ □

2.1.5 Exercise 5

On R^n, define two operations

$$\begin{aligned} \alpha \oplus \beta &= \alpha - \beta \\ c \cdot \alpha &= -c\alpha. \end{aligned}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

Solution. Commutativity of $\oplus$ fails, since $\alpha - \beta$ is not, in general, equal to $\beta - \alpha$. Associativity of $\oplus$ also fails since, for nonzero γ,

$$(\alpha - \beta) - \gamma \neq \alpha - \beta + \gamma = \alpha - (\beta - \gamma).$$

The usual zero vector still works, since $\alpha \oplus 0 = \alpha - 0 = \alpha$. Additive inverses also exist, but they are not the usual ones. Instead, each vector is its own inverse, since $\alpha \oplus \alpha = \alpha - \alpha = 0$.

For multiplication $\cdot$, it is not the case that $1 \cdot \alpha = \alpha$ since, for nonzero α, $1 \cdot \alpha = -1\alpha \neq \alpha$. Associativity with scalar multiplication does not hold either, since

$$(c_1 c_2) \cdot \alpha = -c_1 c_2 \alpha$$

while

$$c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot (-c_2 \alpha) = c_1 c_2 \alpha.$$

For the distributive properties, we have

$$\begin{aligned} c \cdot (\alpha \oplus \beta) &= c \cdot (\alpha - \beta) \\ &= -c(\alpha - \beta) \\ &= -c\alpha + c\beta \end{aligned}$$

and

$$\begin{aligned} c \cdot \alpha \oplus c \cdot \beta &= -c\alpha - (-c\beta) \\ &= -c\alpha + c\beta, \end{aligned}$$

so the first distributive property holds. And

$$\begin{aligned} (c_1 + c_2) \cdot \alpha &= -(c_1 + c_2)\alpha \\ &= -c_1\alpha - c_2\alpha, \end{aligned}$$

while

$$\begin{aligned} c_1 \cdot \alpha \oplus c_2 \cdot \alpha &= -c_1\alpha - (-c_2\alpha) \\ &= -c_1\alpha + c_2\alpha, \end{aligned}$$

so the second distributive property fails.

To summarize, in the vector space definition, only properties (c) and (d) of (3) and (c) of (4) hold. □

2.1.6 Exercise 6

Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$\begin{aligned} (f + g)(t) &= f(t) + g(t) \\ (cf)(t) &= cf(t) \end{aligned}$$

is a vector space over the field of *real* numbers. Give an example of a function in V which is not real-valued.

Solution. Commutativity and associativity of addition follow from the properties of addition in C. Note that the zero function is in V. If $f \in V$, then the function $-f$ given by

$$(-f)(t) = -f(t)$$

is in V since

$$-f(-t) = -\overline{f(t)} = \overline{-f(t)} = \overline{(-f)(t)}.$$

And $f + (-f)$ is the zero function.

For scalar multiplication, we have

$$(1f)(t) = f(t),$$

so the first property is satisfied. And

$$((cd)f)(t) = (cd)f(t) = c((df)(t)) = (c(df))(t),$$

so the second property is satisfied. And distributivity holds, since

$$\begin{aligned}(c(f+g))(t) &= c(f(t)+g(t))\\ &= cf(t)+cg(t)\\ &= (cf)(t)+(cg)(t)\\ &= (cf+cg)(t)\end{aligned}$$

and

$$\begin{aligned}((c+d)f)(t) &= (c+d)f(t)\\ &= cf(t)+df(t)\\ &= (cf)(t)+(df)(t)\\ &= (cf+df)(t).\end{aligned}$$

Therefore V is a vector space over R.

For an example of a function in V, consider the function f from R to C given by

$$f(t) = ti.$$

Then $f(-t) = -ti = \overline{ti} = \overline{f(t)}$ as required. □

2.1.7 Exercise 7

Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$\begin{aligned}(x,y)+(x_1,y_1) &= (x+x_1, 0)\\ c(x,y) &= (cx, 0).\end{aligned}$$

Is V, with these operations, a vector space?

Solution. No, V is not a vector space since $1\alpha = \alpha$ does not hold for all α in V. For example, $1(1,1) = (1,0) \neq (1,1)$. V also fails to have an additive identity. □

2.2 Subspaces

2.2.1 Exercise 1

Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$

Solution. This is not a subspace since it is not closed under scalar multiplication (take any negative scalar). □

(b) all α such that $a_1 + 3a_2 = a_3$

Solution. This is a subspace: Let $\beta = (b_1, b_2, \ldots, b_n)$. Then consider the vector $c\alpha + \beta$. We have

$$\begin{aligned}(ca_1 + b_1) + 3(ca_2 + b_2) &= c(a_1 + 3a_2) + (b_1 + 3b_2)\\ &= ca_3 + b_3.\end{aligned}$$

Therefore $c\alpha + \beta$ is in the subset, so it is a subspace by Theorem 1. □

(c) all α such that $a_2 = a_1^2$

Solution. This is not a subspace since it is not closed under vector addition. For example, $(1, 1, 1, \ldots)$ is in the set, but the sum of this vector with itself is not. □

(d) all α such that $a_1 a_2 = 0$

Solution. This is not a subspace since it is not closed under vector addition. For example $(1, 0, \ldots)$ and $(0, 1, \ldots)$ are each in the set, but their sum is not. □

(e) all α such that a_2 is rational

Solution. This is not a subspace because it is not closed under scalar multiplication: Multiplication of any vector in the set having $a_2 \neq 0$ by an irrational scalar produces a vector that is not in the set. □

2.2.2 Exercise 2

Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

(a) all f such that $f(x^2) = f(x)^2$

Solution. The functions f and g given by

$$f(x) = x \quad \text{and} \quad g(x) = 1$$

each belong to this set, but their sum $f + g$ does not. Therefore this is not a subspace. □

(b) all f such that $f(0) = f(1)$

Solution. Suppose f and g both belong to this set. Then

$$\begin{aligned}(cf+g)(0) &= cf(0)+g(0)\\ &= cf(1)+g(1)\\ &= (cf+g)(1),\end{aligned}$$

so the set satisfies the subspace criterion of Theorem 1 and is thus a subspace of V. □

(c) all f such that $f(3) = 1 + f(-5)$

Solution. Take any f and g in this set. Then

$$(f+g)(3) = f(3) + g(3) = 2 + (f+g)(-5),$$

which does not belong to the set. Therefore this set is not a subspace. □

(d) all f such that $f(-1) = 0$

Solution. Let f and g be such functions. Then

$$(cf+g)(-1) = cf(-1) + g(-1) = 0 + 0 = 0,$$

so this set is a subspace of V by Theorem 1. □

(e) all f which are continuous

Solution. If f and g are continuous, then $cf + g$ is also continuous, so this is a subspace. □

2.2.3 Exercise 3

Is the vector $(3, -1, 0, -1)$ in the subspace of R^5 spanned by the vectors

$$(2, -1, 3, 2), \quad (-1, 1, 1, -3), \quad \text{and} \quad (1, 1, 9, -5)?$$

Solution. The subspace spanned by these three vectors consists of all linear combinations

$$x_1(2, -1, 3, 2) + x_2(-1, 1, 1, -3) + x_3(1, 1, 9, -5).$$

Therefore $(3, -1, 0, -1)$ is in this subspace if and only if the system of equations

$$\begin{aligned}2x_1 - \ x_2 + \ x_3 &= 3\\ -x_1 + \ x_2 + \ x_3 &= -1\\ 3x_1 + \ x_2 + 9x_3 &= 0\\ 2x_1 - 3x_2 - 5x_3 &= -1\end{aligned}$$

has a solution. However, the augmented matrix can be row-reduced to

$$\begin{bmatrix} 2 & -1 & 1 & 3\\ -1 & 1 & 1 & -1\\ 3 & 1 & 9 & 0\\ 2 & -3 & -5 & -1\end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Therefore, this system of equations has no solution and the vector $(3, -1, 0, -1)$ is not in the subspace spanned by the other three given vectors. □

2.2.4 Exercise 4

Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \phantom{{}-3x_5} &= 0 \\ x_1 \phantom{{}-3x_2} + \tfrac{2}{3}x_3 \phantom{{}-3x_4} - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Find a finite set of vectors which spans W.

Solution. After performing the necessary elementary row operations, the coefficient matrix becomes

$$\begin{bmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, letting $x_3 = 3t$, $x_4 = u$, and $x_5 = v$, we see that the elements of W have the form

$$(v - 2t, 2v - u, 3t, u, v).$$

Therefore, a spanning set for W is given by the vectors

$$(-2, 0, 3, 0, 0), \quad (0, -1, 0, 1, 0), \quad \text{and} \quad (1, 2, 0, 0, 1).$$ □

2.2.5 Exercise 5

Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V?

(a) all invertible A

Solution. This cannot be a subspace since the zero matrix is not invertible. □

(b) all non-invertible A

Solution. This is also not a subspace since it is possible for the sum of two non-invertible matrices to be invertible. For example, in the 2×2 case, the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not invertible, but their sum is the identity matrix, which is invertible. □

(c) all A such that $AB = BA$, where B is some fixed matrix in V

Solution. Let A_1 and A_2 be matrices in V such that $A_1B = BA_1$ and $A_2B = BA_2$. Then, since matrix multiplication is distributive,

$$\begin{aligned}(cA_1 + A_2)B &= cA_1B + A_2B \\ &= cBA_1 + BA_2 \\ &= B(cA_1) + BA_2 \\ &= B(cA_1 + A_2).\end{aligned}$$

Therefore, by Theorem 1, this set is a subspace of V. □

(d) all A such that $A^2 = A$

Solution. We will assume that the field F has more than two elements. In that case, this set cannot be a subspace since the identity matrix has the property that $I^2 = I$, but the sum of the identity with itself does not have this property. □

2.2.6 Exercise 6

(a) Prove that the only subspaces of R^1 are R^1 and the zero subspace.

Proof. Suppose W is a subspace of R^1. If $W = \{0\}$ we are done, so suppose W contains a nonzero element x. Then W must contain cx for any real number c. In particular, if r is any real number, then W must contain r since $r = (rx^{-1})x$. This shows that $W = R^1$. □

(b) Prove that a subspace of R^2 is R^2, or the zero subspace, or consists of all scalar multiples of some fixed vector in R^2. (The last type of subspace is, intuitively, a straight line through the origin.)

Proof. Let W be a subspace of R^2. If $W = \{(0, 0)\}$ we are done, so assume W contains a nonzero vector α. Then W must contain all scalar multiples of α. If these are the only elements in W, then we are again finished. If, however, W contains two nonzero elements α and β such that β is *not* a scalar multiple of α, then we must show that $W = R^2$.

Let $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$. Also let $\gamma = (c_1, c_2)$ be any element in R^2. Then γ is a linear combination of α and β if and only if the system of equations

$$\begin{aligned}a_1x_1 + b_1x_2 &= c_1 \\ a_2x_1 + b_2x_2 &= c_2\end{aligned}$$

has a solution. Suppose the coefficient matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is not invertible. By Exercise 1.6.8, we then know that $a_1b_2 - a_2b_1 = 0$. Now, one of a_1 and a_2 is nonzero. If $a_1 \neq 0$, then

$$b_2 = \frac{b_1}{a_1} \cdot a_2.$$

Also

$$b_1 = \frac{b_1}{a_1} \cdot a_1,$$

and we have a contradiction since β was assumed to not be a scalar multiple of α. Similarly $a_2 \neq 0$ also leads to a contradiction. This shows that the system of equations above has a solution, so that $W = R^2$. □

(c) Can you describe the subspaces of R^3?

Solution. The subspaces of R^3 are the zero subspace, the set of all scalar multiples of a fixed nonzero vector (i.e., a line through the origin), the set of all linear combinations of two linearly independent vectors (i.e., a plane through the origin), and R^3 itself. □

2.2.7 Exercise 7

Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

Proof. Let W_1 and W_2 be as stated, but assume that neither is contained in the other. Then there is a vector $u \in W_1$ such that $u \notin W_2$, and there is a vector $v \in W_2$ such that $v \notin W_1$. Since $W_1 \cup W_2$ is a subspace, $u + v \in W_1 \cup W_2$. Now either $u + v \in W_1$ or $u + v \in W_2$. In the first case, since $-u \in W_1$ we must have

$$(u + v) + (-u) = v \in W_1,$$

which is a contradiction. But then $u + v \in W_2$ leads to a similar contradiction. Therefore one of the subspaces W_i must be contained in the other. □

2.2.8 Exercise 8

Let V be the vector space of all functions from R into R; let V_e be the subset of even functions,

$$f(-x) = f(x);$$

let V_o be the subset of odd functions,

$$f(-x) = -f(x).$$

(a) Prove that V_e and V_o are subspaces of V.

Proof. Suppose f and g are even functions. Then for any scalar c,

$$\begin{aligned}(cf + g)(-x) &= cf(-x) + g(-x)\\ &= cf(x) + g(x)\\ &= (cf + g)(x),\end{aligned}$$

so $cf + g$ is also even and therefore V_e is a subspace of V. Similarly, if f and g are both odd functions, then

$$\begin{aligned}(cf + g)(-x) &= cf(-x) + g(-x)\\ &= -cf(x) - g(x)\\ &= -(cf + g)(x),\end{aligned}$$

so V_o is also a subspace. □

(b) Prove that $V_e + V_o = V$.

Proof. Let $f \in V$ be arbitrary. Let g be the function in V defined by

$$g(x) = \frac{f(x) + f(-x)}{2}$$

and let h be the function given by

$$h(x) = \frac{f(x) - f(-x)}{2}.$$

It is clear that $g \in V_e$ and $h \in V_o$. Since $f(x) = g(x) + h(x)$ for all x, we see that $V = V_e + V_o$. □

(c) Prove that $V_e \cap V_o = \{0\}$.

Proof. Suppose $f \in V_e \cap V_o$ and fix a particular $x \in R$. Since f is even, $f(-x) = f(x)$. And since f is odd, $f(-x) = -f(x)$. Therefore we have $f(x) = -f(x)$, which is only possible if $f(x) = 0$. Since x was arbitrary, f must be the zero function. This shows that $V_e \cap V_o$ is the zero subspace. □

2.2.9 Exercise 9

Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 is W_2 such that $\alpha = \alpha_1 + \alpha_2$.

Proof. Since $W_1 + W_2 = V$, we may find α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$. Now suppose there is also α_3 in W_1 and α_4 in W_2 with $\alpha = \alpha_3 + \alpha_4$. Then

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4.$$

Rearranging, we get

$$\alpha_1 - \alpha_3 = \alpha_4 - \alpha_2.$$

But the vector on the left-hand side must belong to W_1, and the vector on the right-hand side must belong to W_2. Therefore $\alpha_1 - \alpha_3$ belongs to the intersection of W_1 and W_2, which implies that $\alpha_1 - \alpha_3 = 0$ or $\alpha_1 = \alpha_3$. And $\alpha_4 = \alpha_2$ also. This shows that the vectors α_1 and α_2 are unique. □

2.3 Bases and Dimension

2.3.1 Exercise 1

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Proof. Let α_1 and α_2 be linearly dependent vectors in the space V. Then, by definition, there are scalars c_1, c_2 not both zero such that

$$c_1\alpha_1 + c_2\alpha_2 = 0.$$

If c_1 is nonzero, then we may write

$$\alpha_1 = -\frac{c_2}{c_1}\alpha_2$$

so that α_1 is a scalar multiple of α_2. If $c_1 = 0$, then c_2 is nonzero and a similar argument will do. □

2.3.2 Exercise 2

Are the vectors

$$\begin{aligned} \alpha_1 &= (1, 1, 2, 4), \quad \alpha_2 = (2, -1, -5, 2) \\ \alpha_3 &= (1, -1, -4, 0), \quad \alpha_4 = (2, 1, 1, 6) \end{aligned}$$

linearly independent in R^4?

Solution. Suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$. This leads to the system of equations

$$\begin{aligned} c_1 + 2c_2 + c_3 + 2c_4 &= 0 \\ c_1 - c_2 - c_3 + c_4 &= 0 \\ 2c_1 - 5c_2 - 4c_3 + c_4 &= 0 \\ 4c_1 + 2c_2 + 6c_4 &= 0. \end{aligned}$$

Using the method of elimination developed in Chapter 1, we find that this system has the general solution

$$(c_1, c_2, c_3, c_4) = \left(\frac{s - 4t}{3}, \frac{-2s - t}{3}, s, t \right),$$

where $s, t \in R^4$ are arbitrary. For example, we may take $s = 3$ and $t = 0$ to get $c_1 = 1$, $c_2 = -2$, $c_3 = 3$, and $c_4 = 0$. This shows that the vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linearly dependent. □

2.3.3 Exercise 3

Find a basis for the subspace of R^4 spanned by the four vectors of Exercise 2.3.2.

Solution. Since α_2 is not a scalar multiple of α_1, the set $\{\alpha_1, \alpha_2\}$ is linearly independent (by Exercise 2.3.1). We also see that it spans the subspace since we can write

$$\alpha_3 = \frac{2}{3}\alpha_2 - \frac{1}{3}\alpha_1$$

and

$$\alpha_4 = \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2.$$

So $\{\alpha_1, \alpha_2\}$ is a basis for the subspace. □

2.3.4 Exercise 4

Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for R^3. Express each of the standard basis vectors as linear combinations of α_1, α_2, and α_3.

Solution. Since $\dim R^3 = 3$, we need only show that the three vectors are independent. Let c_1, c_2, c_3 be scalars such that

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0.$$

Then we arrive at the homogeneous system of equations $Ax = 0$, where A is the 3×3 matrix whose jth column is α_j. By row-reducing this matrix, we see that it is row-equivalent to the identity matrix. Hence the system $Ax = 0$ has only the trivial solution, i.e. $c_1 = c_2 = c_3 = 0$. Therefore $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for R^3.

To write the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$, we may solve the systems $Ax = \epsilon_i$. This gives

$$\begin{aligned}
(1, 0, 0) &= \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \\
(0, 1, 0) &= -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \\
(0, 0, 1) &= -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.
\end{aligned}$$ □

2.3.5 Exercise 5

Find three vectors in R^3 which are linearly dependent, and are such that any two of them are linearly independent.

Solution. Consider the vectors

$$\alpha_1 = (1, 0, 0), \quad \alpha_2 = (0, 1, 0), \quad \text{and} \quad \alpha_3 = (1, 1, 0).$$

These vectors are pairwise-independent since neither is a scalar multiple of another. But they are clearly linearly dependent since $\alpha_1 + \alpha_2 - \alpha_3 = 0$. □

2.3.6 Exercise 6

Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

Proof. We may simply take the standard basis:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Later, in Exercise 2.3.12, we will prove that this set is a basis for V in the more general case where V is the space of $m \times n$ matrices.

Since V has a basis with four elements, it has dimension 4. □

2.3.7 Exercise 7

Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

Proof. Both sets are nonempty. Consider the arbitrary matrices

$$A_1 = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix}$$

in W_1 and let c be an arbitrary scalar. Then

$$cA_1 + A_2 = \begin{bmatrix} cx_1 + x_2 & -cx_1 - x_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix} = \begin{bmatrix} cx_1 + x_2 & -(cx_1 + x_2) \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}$$

which is again in W_1. This shows that W_1 is a subspace of V.

A similar argument will show that W_2 is a subspace of V. □

(b) Find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

Solution. First we find bases for W_1 and W_2. We may take

$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

as a basis for W_1 and

$$\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

as a basis for W_2. Consequently, we see that $\dim W_1 = \dim W_2 = 3$.

Next, observe that matrices in $W_1 \cap W_2$ must have the form

$$\begin{bmatrix} x & -x \\ -x & y \end{bmatrix}.$$

A basis for this space is then

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

so that $\dim(W_1 \cap W_2) = 2$. Finally, we may apply Theorem 6 to determine that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = 3 + 3 - 2 = 4.$$

It follows that $W_1 + W_2 = V$. □

2.3.8 Exercise 8

Again let V be the space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j.

Solution. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

A simple check will show that $A_j^2 = A_j$ for each j. To show that $\{A_1, A_2, A_3, A_4\}$ is a basis for V, we need only show that it spans V (since any spanning set with four vectors must be linearly independent).

Let

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

be an arbitrary 2×2 matrix over F. Then we can write B as a linear combination of A_1, A_2, A_3, A_4 as follows:

$$B = (x - y)A_1 + (w - z)A_2 + yA_3 + zA_4.$$

Therefore the set $\{A_1, A_2, A_3, A_4\}$ is indeed a basis for V. □

2.3.9 Exercise 9

Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent.

Proof. Let c_1, c_2, and c_3 be scalars in F such that

$$c_1(\alpha + \beta) + c_2(\beta + \gamma) + c_3(\gamma + \alpha) = 0.$$

By rearranging, this becomes

$$(c_1 + c_3)\alpha + (c_1 + c_2)\beta + (c_2 + c_3)\gamma = 0.$$

Since α, β, and γ are linearly independent, we must have

$$c_1 + c_3 = 0, \quad c_1 + c_2 = 0, \quad \text{and} \quad c_2 + c_3 = 0.$$

But this system of equations has the unique solution $(c_1, c_2, c_3) = (0, 0, 0)$. Therefore $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent. □

2.3.10 Exercise 10

Let V be a vector space over the field F. Suppose there are a finite number of vectors $\alpha_1, \dots, \alpha_r$ in V which span V. Prove that V is finite-dimensional.

Proof. We know by Theorem 4 that any independent set of vectors in V can have at most a finite number r of elements. Thus, if a basis exists, it must be finite.

We can explicitly construct such a basis: if $\alpha_1, \dots, \alpha_r$ are linearly independent, then we are done. If not, one of the vectors α_i can be written in terms of the other α_j. So remove α_i from the set. This will not affect the span. If the set is now linearly independent, then we have a basis. If not, continue removing elements that are linear combinations of the remaining vectors. This process must eventually terminate since we started with a finite number of vectors in the set. Consequently, a finite basis exists. □

2.3.11 Exercise 11

Let V be the set of all 2×2 matrices A with *complex* entries which satisfy $A_{11} + A_{22} = 0$.

(a) Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

Proof. Let A and B be members of V. Then

$$(A+B)_{11} + (A+B)_{22} = (A_{11} + A_{22}) + (B_{11} + B_{22}) = 0 + 0 = 0.$$

And for any scalar c in R,

$$(cA)_{11} + (cA)_{22} = c(A_{11} + A_{22}) = c0 = 0.$$

This shows that V is closed under matrix addition and scalar multiplication.

Next, we already know that matrix addition is commutative and associative. The zero matrix belongs to V, and for any A in V, the matrix $-A$ is also in V.

The remaining vector space axioms follow from the properties of matrix addition and scalar multiplication. Therefore V is a vector space. □

(b) Find a basis for this vector space.

Solution. One basis is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \right\}.$$

It is fairly straightforward to check that $\mathcal{B}$ both spans V and is linearly independent. □

(c) Let W be the set of all matrices A in V such that $A_{21} = -\overline{A}_{12}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

Solution. First, the zero matrix belongs to W so W is nonempty. Now for any A and B in V and c in R, consider the matrix $cA+B$. We must have

$$(cA+B)_{21} = cA_{21} + B_{21} = -c\overline{A}_{12} - \overline{B}_{12} = -\overline{(cA+B)}_{12}.$$

This shows that W is a subspace of V. A basis for W is given by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}.$$ □

2.3.12 Exercise 12

Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.

Proof. Let $F^{m\times n}$ denote the space of $m \times n$ matrices over F.

For each i and j with $1 \le i \le m$ and $1 \le j \le n$, let ϵ_{ij} denote the $m \times n$ matrix over F whose ijth entry is 1, with all other entries 0. Let $\mathcal{B}$ denote the set of all ϵ_{ij}. We will show that $\mathcal{B}$ is a basis for $F^{m\times n}$, so that the dimension of this space is mn.

First, let

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\epsilon_{ij},$$

where each c_{ij} is an arbitrary scalar in F. Then A is the matrix whose ijth entry is c_{ij}. By choosing these scalars appropriately, we see that any $m \times n$ matrix over F can be written as a linear combination of the matrices in $\mathcal{B}$. Therefore $\mathcal{B}$ spans $F^{m\times n}$.

Moreover, $A = 0$ if and only if each $c_{ij} = 0$, so $\mathcal{B}$ is linearly independent. This shows that $\mathcal{B}$ is a basis for $F^{m\times n}$. □

2.3.13 Exercise 13

Discuss Exercise 2.3.9, when V is a vector space over the field with two elements described in Exercise 1.2.5.

Solution. In Exercise 2.3.9, it was stated that the field F should be a subfield of the complex numbers (in particular, a field with characteristic 0). When this restriction is taken away, the result does not necessarily hold, as we will now demonstrate.

Let V be the vector space F^3, where F is the field with 2 elements. Let $\alpha = (1,0,0)$, $\beta = (0,1,0)$, and $\gamma = (0,0,1)$. We see that α, β, and γ are linearly independent (in fact they form the standard basis of F^3).

Now consider the vectors

$$\alpha + \beta = (1,1,0), \quad \beta + \gamma = (0,1,1), \quad \text{and} \quad \gamma + \alpha = (1,0,1).$$

These are *not* linearly independent, since

$$(1,1,0) + (0,1,1) + (1,0,1) = (0,0,0).$$

So the result from Exercise 2.3.9 does not hold in this more general setting. □

2.3.14 Exercise 14

Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Proof. Assume the contrary, and let $\{x_1, x_2, \ldots, x_n\}$ be a finite basis for V. Then every real number can be expressed as a linear combination

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where $c_1, \ldots, c_n$ are rational numbers. Thus we can establish a one-to-one correspondence between the n-tuples of rational numbers and the set of real numbers. Since the rational numbers are countable, this implies that the reals are also countable. But this is clearly a contradiction. Therefore V is *not* finite-dimensional. □

2.4 Coordinates

2.4.1 Exercise 1

Show that the vectors

$$\alpha_1 = (1,1,0,0), \quad \alpha_2 = (0,0,1,1)$$
$$\alpha_3 = (1,0,0,4), \quad \alpha_4 = (0,0,0,2)$$

form a basis for R^4. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Solution. Let

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}.$$

P is invertible and has inverse

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

By Theorem 8, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis for R^4. Moreover, the jth column of P^{-1} gives the coordinates of the standard basis vector ϵ_j in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. □

2.4.2 Exercise 2

Find the coordinate matrix of the vector $(1,0,1)$ in the basis of C^3 consisting of the vectors $(2i,1,0)$, $(2,-1,1)$, $(0,1+i,1-i)$, in that order.

Solution. Let

$$P = \begin{bmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ -\frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{bmatrix}.$$

Since

$$P^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ -\frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2}i \\ \frac{3}{4} + \frac{1}{4}i \end{bmatrix},$$

the vector $(1,0,1)$ has coordinates $\left(-\frac{1}{2} - \frac{1}{2}i, \frac{1}{2}i, \frac{3}{4} + \frac{1}{4}i\right)$ in the given basis. □

2.4.3 Exercise 3

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for R^3 consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1) \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

Solution. Let

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

and

$$P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - c \\ b \\ a - 2b + c \end{bmatrix}.$$

So (a, b, c) has coordinates $(b - c, b, a - 2b + c)$ in the ordered basis $\mathcal{B}$. □

2.4.4 Exercise 4

Let W be the subspace of C^3 spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W.

Solution. Since neither α_1 nor α_2 is a scalar multiple of the other, the set $\{\alpha_1, \alpha_2\}$ is linearly independent. Hence this set is a basis for W. □

(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W.

Solution. If $c_1(1, 0, i) + c_2(1 + i, 1, -1) = (1, 1, 0)$, then equating coordinates and solving the resulting system gives $c_1 = -i$ and $c_2 = 1$. Therefore β_1 is in W and its coordinates in $\{\alpha_1, \alpha_2\}$ are $(-i, 1)$.

In a similar way, we can determine that β_2 is in W and has coordinates $(2 - i, i)$ in the given basis.

Neither β_1 nor β_2 is a scalar multiple of the other, so the set $\{\beta_1, \beta_2\}$ is linearly independent. Since W has dimension 2, the set $\{\beta_1, \beta_2\}$ is also a basis for W. □

(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?

Solution. From the coordinates for β_1 and β_2 that we found previously, we get the transition matrix

$$P = \begin{bmatrix} -i & 2-i \\ 1 & i \end{bmatrix}.$$

This matrix has inverse

$$P^{-1} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & \frac{3}{2} + \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i & -\frac{1}{2} + \frac{1}{2}i \end{bmatrix},$$

so

$$\alpha_1 = \left(\frac{1}{2} - \frac{1}{2}i\right)\beta_1 + \left(\frac{1}{2} + \frac{1}{2}i\right)\beta_2$$

and

$$\alpha_2 = \left(\frac{3}{2} + \frac{1}{2}i\right)\beta_1 + \left(-\frac{1}{2} + \frac{1}{2}i\right)\beta_2.$$

□

2.4.6 Exercise 6

Let V be the vector space over the complex numbers of all functions from R into C, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.

Proof. Let c_1, c_2, and c_3 be complex numbers such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all x in R. Then

$$c_1 + c_2 e^{ix} + c_3 e^{-ix} = 0.$$

Using Euler's formula, we can write

$$c_1 + c_2(\cos x + i \sin x) + c_3(\cos x - i \sin x) = 0$$

or, rearranging,

$$c_1 + (c_2 + c_3)\cos x + (c_2 - c_3) i \sin x = 0. \tag{2.1}$$

If $x = 0$, then

$$c_1 + c_2 + c_3 = 0, \tag{2.2}$$

while if $x = \pi$ we get

$$c_1 - c_2 - c_3 = 0. \tag{2.3}$$

Equations (2.2) and (2.3) together imply that $c_1 = 0$.

Next, letting $x = \pi/2$ in (2.1), we get

$$(c_2 - c_3) i = 0, \tag{2.4}$$

which implies that $c_2 = c_3$. Equation (2.2) then implies that $c_2 = c_3 = 0$.

Since it is necessary that $c_1 = c_2 = c_3 = 0$, it follows that $\{f_1, f_2, f_3\}$ is a linearly independent set. □

(b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^{3} P_{ij} f_i.$$

Solution. First, we have $g_1 = f_1$. Next, since

$$f_2(x) + f_3(x) = (\cos x + i \sin x) + (\cos x - i \sin x) = 2 \cos x,$$

we have $g_2 = \frac{1}{2} f_2 + \frac{1}{2} f_3$. And since

$$f_2(x) - f_3(x) = 2i \sin x,$$

we see that $g_3 = -\frac{1}{2} i f_2 + \frac{1}{2} i f_3$. Therefore the desired matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} i \\ 0 & \frac{1}{2} & \frac{1}{2} i \end{bmatrix},$$

and this matrix is invertible. □

2.4.7 Exercise 7

Let V be the (real) vector space of all polynomial functions from R into R of degree 2 or less, i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x + t)^2.$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in this ordered basis $\mathcal{B}$?

Solution. Let

$$A = a + b(x + t) + c(x + t)^2 = (a + bt + ct^2) + (b + 2ct)x + cx^2,$$

where a, b, c are real numbers. First, if $A = 0$ then, by equating coefficients, we have

$$\begin{aligned} a + bt + ct^2 &= 0, \\ b + 2ct &= 0, \\ c &= 0. \end{aligned}$$

Working backward through the equations, we see that a, b, and c must all be 0. This shows that $\mathcal{B}$ is linearly independent.

If we now set $A = c_0 + c_1x + c_2x^2$, where c_1, c_2, c_3 are arbitrary, and equate coefficients, we get

$$\begin{aligned} a + bt + ct^2 &= c_0, \\ b + 2ct &= c_1, \\ c &= c_2. \end{aligned}$$

Through back-substitution, we find that

$$b = c_1 - 2tc_2 \quad \text{and} \quad a = c_0 - tc_1 + t^2c_2.$$

This shows that any polynomial of degree 2 or less can be written as a linear combination of g_1, g_2, and g_3. $\mathcal{B}$ is therefore a basis for V.

Moreover, we have also shown that the polynomial $f(x) = c_0 + c_1x + c_2x^2$ has coordinates

$$(c_0 - tc_1 + t^2c_2, c_1 - 2tc_2, c_2)$$

in the ordered basis $\{g_1, g_2, g_3\}$. $\square$

2.6 Computations Concerning Subspaces

2.6.1 Exercise 1

Let $s < n$ and A an $s \times n$ matrix with entries in the field F. Use Theorem 4 (not its proof) to show that there is a non-zero X in $F^{n\times 1}$ such that $AX = 0$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ denote the columns of A. Then each α_i is a member of the vector space F^s, which has dimension strictly less than n. Therefore, by Theorem 4, the α_i are necessarily linearly dependent. Thus we can write

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

for $c_1, \ldots, c_n$ in F not all 0. If we let

$$X = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

then $AX = 0$ as required. □

2.6.2 Exercise 2

Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

(a) Which of the vectors α, β, γ are in the subspace of R^4 spanned by the α_i?

Solution. Let A be the 3×4 matrix whose ith row is α_i. By performing row reduction on A, we get

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 0 & 4 & -1 \\ -1 & 2 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3/13 \\ 0 & 1 & 0 & 14/13 \\ 0 & 0 & 1 & -1/13 \end{bmatrix}.$$

A vector ρ is in the row space of A if and only if

$$\rho = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \left(c_1, c_2, c_3, -\frac{3}{13}c_1 + \frac{14}{13}c_2 - \frac{1}{13}c_3\right).$$

Checking each of α, β, and γ, we see that only α is in the row space. So α is in the subspace spanned by the α_i, while β and γ are not in the subspace. □

(b) Which of the vectors α, β, γ are in the subspace of C^4 spanned by the α_i?

Solution. Our work above is still valid in C^4. α_1, α_2, and α_3 will span a larger subspace due to the scalars being taken from C instead of R, but the members of this subspace will still have the same form as before. Thus, of the three vectors α, β, and γ, only α is in the subspace. □

(c) Does this suggest a theorem?

Solution. This suggests the following theorem: let F be a subfield of the field E. Let α be a vector in F^n, and let $\beta_1, \ldots, \beta_n$ in F^n span some subspace. Then α is in this subspace of F^n if and only if it is in the subspace of E^n spanned by the same vectors β_i. □

2.6.3 Exercise 3

Consider the vectors in R^4 defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of R^4 spanned by the three given vectors.

Solution. Let A be the 3×4 matrix whose ith row is α_i. We can perform row-reduction on A to get

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then a vector ρ in R^4 is in the row space of A if and only if it has the form

$$\rho = \left(r_1, r_2, \frac{1}{4} r_2 - r_1, \frac{11}{4} r_2 - 2r_1 \right),$$

where r_1 and r_2 are real numbers. If we label the components of ρ as

$$\rho = (x_1, x_2, x_3, x_4),$$

then we get the following system of equations:

$$\begin{aligned} x_3 &= \frac{1}{4} x_2 - x_1 \\ x_4 &= \frac{11}{4} x_2 - 2x_1. \end{aligned}$$

Or, we can rearrange these equations to write

$$\begin{aligned} x_1 - \frac{1}{4} x_2 + x_3 \qquad &= 0 \\ 2x_1 - \frac{11}{4} x_2 \qquad + x_4 &= 0. \end{aligned}$$

This system of equations is homogeneous and its solution set is precisely the subspace spanned by α_1, α_2, and α_3. □

2.6.4 Exercise 4

In C^3, let

$$\alpha_1 = (1, 0, -i), \quad \alpha_2 = (1+i, 1-i, 1), \quad \alpha_3 = (i, i, i).$$

Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a, b, c) in this basis?

Solution. Let

$$A = \begin{bmatrix} 1 & 0 & -i \\ 1+i & 1-i & 1 \\ i & i & i \end{bmatrix}.$$

By performing row-reduction, one can verify that A is row-equivalent to the identity matrix. So A has rank 3 and α_1, α_2, and α_3 are linearly independent and span C^3, as required to be a basis.

Let the coordinates of (a, b, c) in this basis be (x, y, z). This leads to the following system of equations.

$$\begin{aligned} x + (1+i)y + iz &= a \\ (1-i)y + iz &= b \\ -ix + \qquad y + iz &= c. \end{aligned}$$

With a bit of effort, one may determine this system to have the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a+b-2c}{5} + \frac{4c-2a-2b}{5} i \\ \frac{a+b-2c}{5} + \frac{3b-2a-c}{5} i \\ \frac{3a-2b-c}{5} - \frac{a+b+3c}{5} i \end{bmatrix}.$$

□

2.6.5 Exercise 5

Give an explicit description for the vectors

$$\beta = (b_1, b_2, b_3, b_4, b_5)$$

in R^5 which are linear combinations of the vectors

$$\begin{aligned} \alpha_1 &= (1, 0, 2, 1, -1), & \alpha_2 &= (-1, 2, -4, 2, 0) \\ \alpha_3 &= (2, -1, 5, 2, 1), & \alpha_4 &= (2, 1, 3, 5, 2). \end{aligned}$$

Solution. Performing row-reduction on the augmented matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 & b_1 \\ 0 & 2 & -1 & 1 & b_2 \\ 2 & -4 & 5 & 3 & b_3 \\ 1 & 2 & 2 & 5 & b_4 \\ -1 & 0 & 1 & 2 & b_5 \end{bmatrix}$$

produces

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{3}b_1 + \frac{1}{2}b_2 - \frac{1}{6}b_4 - \frac{1}{2}b_5 \\ 0 & 1 & 0 & 0 & \frac{7}{6}b_4 - \frac{5}{3}b_1 - \frac{3}{2}b_2 - \frac{1}{2}b_5 \\ 0 & 0 & 1 & 0 & \frac{3}{2}b_4 - 2b_1 - \frac{5}{2}b_2 - \frac{1}{2}b_5 \\ 0 & 0 & 0 & 1 & \frac{4}{3}b_1 + \frac{3}{2}b_2 - \frac{5}{6}b_4 + \frac{1}{2}b_5 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.$$

Thus we see that $\beta = (b_1, b_2, b_3, b_4, b_5)$ is in the subspace spanned by α_1, α_2, α_3, α_4 if and only if $b_3 + b_2 - 2b_1 = 0$. For any such β, we have

$$\begin{aligned}\beta = &\left(\frac{2}{3}b_1 + \frac{1}{2}b_2 - \frac{1}{6}b_4 - \frac{1}{2}b_5\right)\alpha_1 + \left(\frac{7}{6}b_4 - \frac{5}{3}b_1 - \frac{3}{2}b_2 - \frac{1}{2}b_5\right)\alpha_2 \\ &+ \left(\frac{3}{2}b_4 - 2b_1 - \frac{5}{2}b_2 - \frac{1}{2}b_5\right)\alpha_3 + \left(\frac{4}{3}b_1 + \frac{3}{2}b_2 - \frac{5}{6}b_4 + \frac{1}{2}b_5\right)\alpha_4.\end{aligned}$$

□

2.6.6 Exercise 6

Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}.$$

(a) Find a basis for V.

Solution. If we perform row-reduction on the matrix A, we get the row-reduced echelon matrix

$$R = \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The three nonzero rows ρ_1, ρ_2, and ρ_3 of R form a basis for V. □

(b) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.

Solution. If we take linear combinations of ρ_1, ρ_2, and ρ_3, we can see that the vector $(x_1, x_2, x_3, x_4, x_5)$ is in V if and only if

$$x_2 = 7x_1 \quad \text{and} \quad x_4 = 3x_1 + 5x_3.$$ □

(c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?

Solution. Let $x = (x_1, x_2, x_3, x_4, x_5)$ be in V. If

$$x = c_1\rho_1 + c_2\rho_2 + c_3\rho_3,$$

then we get the following system of equations:

$$\begin{aligned} c_1 &= x_1 \\ 7c_1 &= x_2 \\ c_2 &= x_3 \\ 3c_1 + 5c_2 &= x_4 \\ c_3 &= x_5. \end{aligned}$$

We see that x has coordinates (x_1, x_3, x_5) in the basis $\{\rho_1, \rho_2, \rho_3\}$. □

2.6.7 Exercise 7

Let A be an $m \times n$ matrix over the field F, and consider the system of equations $AX = Y$. Prove that this system of equations has a solution if and only if the row rank of A is equal to the row rank of the augmented matrix of the system.

Proof. Let R be the row-reduced echelon matrix that is row-equivalent to A. Form the augmented matrix A' and let R' be the row-reduced echelon matrix row-equivalent to A'. Then the nonzero rows of R form a basis for the row space of A, and the nonzero rows of R' form a basis for the row space of A'. We want to show that these bases have the same number of elements.

By the nature of the process of row reduction, it must be that the first n columns of R' will be identical to the n columns of R. Consequently, R' cannot have fewer nonzero rows than R, as any nonzero row of R must correspond to a nonzero row in R'. However, it might be possible for R' to have more nonzero rows than R. Such nonzero rows would need to have zeros in every column except the last. But then such a row would indicate that the system $AX = Y$ has no solutions, which we know to be false. Therefore A and A' have the same row rank.

Now let us consider the converse. Let R and R' be as before, and suppose that the row ranks of A and A' are equal. If $AX = Y$ has no solutions, then R' would necessarily have a row consisting of zeros in every column but the last. But then the corresponding row in R would have only zero entries, resulting in R' having a larger row rank than R. This is impossible, so $AX = Y$ must have a solution. □

Chapter 3

Linear Transformations

3.1 Linear Transformations

3.1.1 Exercise 1

Which of the following functions T from R^2 into R^2 are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$;

Solution. T cannot be a linear transformation since

$$T(0,0) = (1,0) \neq (0,0). \qquad \square$$

(b) $T(x_1, x_2) = (x_2, x_1)$;

Solution. T is a linear transformation: let c be a scalar and let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Then

$$\begin{aligned} T(c\alpha + \beta) &= T(cx_1 + y_1, cx_2 + y_2) \\ &= (cx_2 + y_2, cx_1 + y_1) \\ &= c(x_2, x_1) + (y_2, y_1) \\ &= cT(\alpha) + T(\beta). \end{aligned} \qquad \square$$

(c) $T(x_1, x_2) = (x_1^2, x_2)$;

Solution. T is not a linear transformation. For example,

$$T(1,0) + T(2,0) = (1,0) + (4,0) = (5,0)$$

but

$$T((1,0) + (2,0)) = T(3,0) = (9,0). \qquad \square$$

(d) $T(x_1, x_2) = (\sin x_1, x_2)$;

Solution. T is not a linear transformation since

$$T\left(\frac{\pi}{2},0\right)+T\left(\frac{\pi}{2},0\right)=(1,0)+(1,0)=(2,0)$$

while

$$T\left(\left(\frac{\pi}{2},0\right)+\left(\frac{\pi}{2},0\right)\right)=T(\pi,0)=(0,0).$$ □

(e) $T(x_1,x_2)=(x_1-x_2,0)$.

Solution. Let $\alpha=(x_1,x_2)$ and $\beta=(y_1,y_2)$. Since

$$\begin{aligned}cT(\alpha)+T(\beta)&=c(x_1-x_2,0)+(y_1-y_2,0)\\&=(cx_1-cx_2+y_1-y_2,0)\\&=((cx_1+y_1)-(cx_2+y_2),0)\\&=T(cx_1+y_1,cx_2+y_2)\\&=T(c\alpha+\beta),\end{aligned}$$

we see that T is a linear transformation. □

3.1.2 Exercise 2

Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.

Solution. Let V be a vector space of finite dimension n, let $T\colon V\to V$ be the zero transformation, and let $U\colon V\to V$ be the identity transformation.

Then the range of T is clearly the set consisting of the zero vector alone, and the null space is V itself. The rank of T is then 0 (the zero subspace has the empty set as a basis) and the nullity of T is n.

For the identity transformation U, we see that the range is all of V, while the null space is the zero subspace $\{0\}$. Then the rank of U is n and the nullity of U is 0.

Note that in each case, the rank plus the nullity is n, in agreement with Theorem 2. □

3.1.3 Exercise 3

Describe the range and the null space for the differentiation transformation of Example 2. Do the same for the integration transformation of Example 5.

Solution. Let V be the space of polynomial functions from F into F and let D be the differentiation transformation. Given a polynomial

$$f(x)=c_0+c_1x+\cdots+c_kx^k,$$

we can always find another polynomial $g(x)$ such that $(Dg)(x)=f(x)$, namely the polynomial

$$g(x)=c_0x+\frac{1}{2}c_1x^2+\cdots+\frac{1}{k+1}c_kx^{k+1}.$$

Therefore the range of D is V.

A function f has zero derivative if and only if f is constant,

$$f(x) = c_0, \quad \text{for some } c_0 \text{ in } F.$$

So the null space of D is the space of constant functions.

Now, let T be the integration transformation defined in Example 5. We can integrate any polynomial to get another polynomial, but because of the manner in which T was defined, the resulting polynomial will always have a constant term of 0. Let f be a polynomial with constant term zero,

$$f(x) = c_1x + c_2x^2 + \cdots + c_kx^k.$$

Then the function $g(x) = (Df)(x)$ is such that $(Tg)(x) = f(x)$. So we see that the range of T is the space of polynomials with zero constant term.

Lastly, if f is a polynomial such that $(Tf)(x) = 0$, then f must be the zero polynomial. The null space of T is therefore the trivial subspace $\{0\}$. □

3.1.4 Exercise 4

Is there a linear transformation T from R^3 into R^2 such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$?

Solution. Yes. In fact, there are infinitely many such transformations, as we will now show.

Let $\alpha = (1, -1, 1)$ and let $\beta = (1, 1, 1)$. Since neither α nor β is a multiple of the other, the set $\{\alpha, \beta\}$ is linearly independent. Therefore it can be extended to a basis for R^3. The existence of a linear transformation T such that $T(\alpha) = (1, 0)$ and $T(\beta) = (0, 1)$ now follows from Theorem 1.

To find such a transformation explicitly, we will form a basis for R^3. If we let $\gamma = (1, 0, 0)$, for example, it is not difficult to show that $\{\alpha, \beta, \gamma\}$ is a linearly independent set of vectors which spans R^3.

We want to be able to write a vector $\rho = (b_1, b_2, b_3)$ as a linear combination of α, β, and γ. To do this, we will set up an augmented matrix and perform row-reduction, as we have done before in Chapter 2:

$$\begin{bmatrix} 1 & 1 & 1 & y_1 \\ -1 & 1 & 0 & y_2 \\ 1 & 1 & 0 & y_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}y_3 - \frac{1}{2}y_2 \\ 0 & 1 & 0 & \frac{1}{2}y_2 + \frac{1}{2}y_3 \\ 0 & 0 & 1 & y_1 - y_3 \end{bmatrix}.$$

So we may write

$$\rho = (b_1, b_2, b_3) = \frac{1}{2}(b_3 - b_2)\alpha + \frac{1}{2}(b_2 + b_3)\beta + (b_1 - b_3)\gamma.$$

Now suppose the transformation T is such that $T(\gamma) = (x_1, x_2)$, for some x_1

and x_2 in R. Then we have

$$\begin{aligned} T(b_1,b_2,b_3) &= T\left(\frac{1}{2}(b_3-b_2)\alpha + \frac{1}{2}(b_2+b_3)\beta + (b_1-b_3)\gamma\right) \\ &= \frac{1}{2}(b_3-b_2)T(\alpha) + \frac{1}{2}(b_2+b_3)T(\beta) + (b_1-b_3)T(\gamma) \\ &= \frac{1}{2}(b_3-b_2,0) + \frac{1}{2}(0,b_2+b_3) + (x_1b_1-x_1b_3, x_2b_1-x_2b_3) \\ &= \frac{1}{2}(2x_1b_1 - b_2 + (1-2x_1)b_3, 2x_2b_1 + b_2 + (1-2x_2)b_3). \end{aligned}$$

So, for example, taking $(x_1,x_2) = \left(\frac{1}{2},\frac{1}{2}\right)$, we get

$$T(b_1,b_2,b_3) = \left(\frac{1}{2}b_1 - \frac{1}{2}b_2, \frac{1}{2}b_1 + \frac{1}{2}b_2\right).$$

By picking different values for x_1 and x_2, we see that there are infinitely many possibilities for T. □

3.1.5 Exercise 5

If

$$\begin{aligned} \alpha_1 &= (1,-1), \quad & \beta_1 &= (1,0) \\ \alpha_2 &= (2,-1), \quad & \beta_2 &= (0,1) \\ \alpha_3 &= (-3,2), \quad & \beta_3 &= (1,1) \end{aligned}$$

is there a linear transformation T from R^2 into R^2 such that $T\alpha_i = \beta_i$ for $i = 1$, 2, and 3?

Solution. No. To see why, observe that

$$(-3,2) = -(1,-1) - (2,-1).$$

We have

$$T(-\alpha_1 - \alpha_2) = (1,1)$$

but

$$-T(\alpha_1) - T(\alpha_2) = -(1,0) - (0,1) = (-1,-1).$$

So T cannot be a linear transformation. □

3.1.6 Exercise 6

Describe explicitly the linear transformation T from F^2 into F^2 such that

$$T\epsilon_1 = (a,b), \quad T\epsilon_2 = (c,d).$$

Solution. For any (x_1,x_2) in F^2, we have

$$\begin{aligned} T(x_1,x_2) &= T(x_1\epsilon_1 + x_2\epsilon_2) \\ &= x_1T(\epsilon_1) + x_2T(\epsilon_2) \\ &= x_1(a,b) + x_2(c,d) \\ &= (x_1a + x_2c, x_1b + x_2d). \end{aligned}$$

□

3.1.7 Exercise 7

Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

(a) Verify that T is a linear transformation.

Solution. Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. Also for x in F^3, let $\pi_1(x)$ denote the first coordinate of x, $\pi_2(x)$ the second coordinate, and $\pi_3(x)$ the third coordinate. Then

$$\begin{aligned}
\pi_1(cT(\alpha) + T(\beta)) &= c(x_1 - x_2 + 2x_3) + (y_1 - y_2 + 2y_3) \\
&= (cx_1 + y_1) - (cx_2 + y_2) + 2(cx_3 + y_3) \\
&= \pi_1(T(c\alpha + \beta)), \\
\pi_2(cT(\alpha) + T(\beta)) &= c(2x_1 + x_2) + (2y_1 + y_2) \\
&= 2(cx_1 + y_1) + (cx_2 + y_2) \\
&= \pi_2(T(c\alpha + \beta)),
\end{aligned}$$

and

$$\begin{aligned}
\pi_3(cT(\alpha) + T(\beta)) &= c(-x_1 - 2x_2 + 2x_3) + (-y_1 - 2y_2 + 2y_3) \\
&= -(cx_1 + y_1) - 2(cx_2 + y_2) + 2(cx_3 + y_3) \\
&= \pi_3(T(c\alpha + \beta)).
\end{aligned}$$

This shows that T is a linear transformation. $\square$

(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b, and c that the vector be in the range of T? What is the rank of T?

Solution. If (a, b, c) is in the range of T, then

$$\begin{aligned}
x_1 - x_2 + 2x_3 &= a \\
2x_1 + x_2 &= b \\
-x_1 - 2x_2 + 2x_3 &= c.
\end{aligned}$$

In performing row-reduction on the augmented matrix for the above system, we get

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{1}{3}a + \frac{1}{3}b \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3}a + \frac{1}{3}b \\ 0 & 0 & 0 & -a + b + c \end{bmatrix}.$$

From this latter matrix, we see that this system of equations has a solution if and only if

$$-a + b + c = 0.$$

We also see that the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

has a row rank (and thus column rank) of 2. But the column space of A is precisely the range of T, so we may conclude that T has rank 2. □

(c) What are the conditions on a, b, and c that (a, b, c) be in the null space of T? What is the nullity of T?

Solution. (a, b, c) is in the null space of T if and only if

$$\begin{aligned} a - b + 2c &= 0 \\ 2a + b \phantom{{}+2c} &= 0 \\ -a - 2b + 2c &= 0. \end{aligned}$$

We have already seen above that the coefficient matrix reduces to

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

So (a, b, c) is in the null space if and only if $a + 2c/3 = 0$ and $b - 4c/3 = 0$.

Letting $c = -3$, for example, we find one possible basis for the null space of T to be $\{(2, -4, -3)\}$. We see that the nullity of T is therefore 1, which is as we should expect since F^3 has dimension 3 and the rank of T is 2. □

3.1.8 Exercise 8

Describe explicitly a linear transformation from R^3 into R^3 which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

Solution. Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ denote the standard ordered basis for R^3. Theorem 1 allows us to find infinitely many linear transformations satisfying the given criterion. For example, we may take some linear combination of the two given vectors, say $(2, 2, 1)$, and then look for a linear transformation T such that

$$T\epsilon_1 = (1, 0, -1), \quad T\epsilon_2 = (1, 2, 2), \quad \text{and} \quad T\epsilon_3 = (2, 2, 1).$$

Evidently, the transformation

$$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, 2x_2 + 2x_3, -x_1 + 2x_2 + x_3)$$

does the job. The range of T is precisely the subspace of R^3 spanned by $(1, 0, -1)$ and $(1, 2, 2)$. Of course, as noted, there are infinitely many other transformations that would work. □

3.1.9 Exercise 9

Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V.

Proof. Let A_1 and A_2 be members of V and let c be a scalar in F. Using the ordinary properties of matrix addition and multiplication, we have

$$\begin{aligned} cT(A_1) + T(A_2) &= c(A_1B - BA_1) + (A_2B - BA_2) \\ &= (cA_1 + A_2)B - B(cA_1 + A_2) \\ &= T(cA_1 + A_2). \end{aligned}$$

Therefore T is a linear transformation from V into V. □

3.1.10 Exercise 10

Let V be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on C^1, i.e., which is not complex linear.

Solution. Define $T(a + bi) = a$. Then for any real number c and any complex numbers $z = a_1 + b_1 i$ and $w = a_2 + b_2 i$, we have

$$cT(z) + T(w) = ca_1 + a_2 = T(cz + w),$$

so T is a linear transformation from V into V. However,

$$iT(1) = i \neq 0 = T(i),$$

so T is *not* linear on C^1. □

3.1.11 Exercise 11

Let V be the space of $n \times 1$ matrices over F and let W be the space of $m \times 1$ matrices over F. Let A be a fixed $m \times n$ matrix over F and let T be the linear transformation from V into W defined by $T(X) = AX$. Prove that T is the zero transformation if and only if A is the zero matrix.

Proof. First suppose that T is the zero transformation. Then $AX = 0$ for all X in V. In particular, let $X = \epsilon_j$, where ϵ_j is the column vector whose jth entry is 1 and all other entries zero. Then $AX = 0$ implies that the jth column of A has only zero entries. Since this is true for all j with $1 \leq j \leq n$, we see that A is the $m \times n$ zero matrix.

Conversely, let A be the zero matrix. Then $AX = 0$ for all X so T is clearly the zero transformation. □

3.1.12 Exercise 12

Let V be an n-dimensional vector space over the field F and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove that n is even. (Can you give an example of such a linear transformation T?)

Solution. This result follows directly from Theorem 2: if the rank of T is k, then the nullity is also k and we have

$$k + k = n,$$

or $n = 2k$. Hence n is even.

As an example, let $V = R^2$ and consider the linear transformation T given by

$$T(x, y) = (y, 0).$$

Since

$$\begin{aligned} cT(x_1, y_1) + T(x_2, y_2) &= c(y_1, 0) + (y_2, 0) \\ &= (cy_1 + y_2, 0) \\ &= T(c(x_1, y_1) + (x_2, y_2)), \end{aligned}$$

T is a linear transformation. And both the range and the null space of T is the x-axis. □

3.1.13 Exercise 13

Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.

(a) The intersection of the range of T and the null space of T is the zero subspace of V.

(b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

Proof. Assume that (a) is true. If $T(T\alpha) = 0$, then $T\alpha$ belongs to the null space of T. But $T\alpha$ is also in the range of T, so $T\alpha = 0$ by assumption.

Conversely, assume that (b) holds. Let β belong to the intersection of the range of T with the null space of T. Then $T(\beta) = 0$ and there is some α in V such that $T(\alpha) = \beta$. Then $T(T\alpha) = T(\beta) = 0$, so that $T\alpha = 0$ by assumption. But $T\alpha = \beta$, so $\beta = 0$. Therefore the specified intersection is the zero subspace and (a) holds. □

3.2 The Algebra of Linear Transformations

3.2.1 Exercise 1

Let T and U be the linear operators on R^2 defined by

$$T(x_1, x_2) = (x_2, x_1) \quad \text{and} \quad U(x_1, x_2) = (x_1, 0).$$

(a) How would you describe T and U geometrically?

Solution. T mirrors points across the line $y = x$. U projects points onto the x-axis. □

(b) Give rules like the ones defining T and U for each of the transformations $(U + T)$, UT, TU, T^2, U^2.

Solution. We have

$$\begin{aligned}
(U + T)(x_1, x_2) &= (x_1 + x_2, x_1),\\
UT(x_1, x_2) &= (x_2, 0),\\
TU(x_1, x_2) &= (0, x_1),\\
T^2(x_1, x_2) &= (x_1, x_2),
\end{aligned}$$

and

$$U^2(x_1, x_2) = (x_1, 0).$$ □

3.2.2 Exercise 2

Let T be the (unique) linear operator on C^3 for which

$$T\epsilon_1 = (1, 0, i), \quad T\epsilon_2 = (0, 1, 1), \quad T\epsilon_3 = (i, 1, 0).$$

Is T invertible?

Solution. Let $\alpha = (z_1, z_2, z_3)$ be a vector in C^3 such that $T\alpha = 0$. Then

$$z_1(1, 0, i) + z_2(0, 1, 1) + z_3(i, 1, 0) = (0, 0, 0),$$

or

$$\begin{aligned}
z_1 \qquad + z_3 i &= 0\\
z_2 + \ z_3 &= 0\\
z_1 i + z_2 \qquad &= 0.
\end{aligned}$$

This system of equations has infinitely many solutions, each of the form

$$(z_1, z_2, z_3) = (-ti, -t, t).$$

Therefore the null space of T is not $\{0\}$, so T is singular and *not* invertible. □

3.2.3 Exercise 3

Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one which defines T.

Solution. It is not difficult to see that T is non-singular since

$$T(x_1, x_2, x_3) = (0, 0, 0) \quad \text{if and only if} \quad x_1 = x_2 = x_3 = 0.$$

By Theorem 9, T is invertible.

Let $\alpha = (y_1, y_2, y_3)$ in R^3 be such that $T\alpha = (x_1, x_2, x_3)$. Then

$$\begin{aligned} 3y_1 \phantom{{}+y_2+y_3} &= x_1 \\ y_1 - y_2 \phantom{{}+y_3} &= x_2 \\ 2y_1 + y_2 + y_3 &= x_3. \end{aligned}$$

We see that this system of equations has the unique solution

$$(y_1, y_2, y_3) = \left(\frac{1}{3}x_1, \frac{1}{3}x_1 - x_2, x_3 - x_1 + x_2\right).$$

So we have

$$T^{-1}(x_1, x_2, x_3) = \left(\frac{1}{3}x_1, \frac{1}{3}x_1 - x_2, x_3 - x_1 + x_2\right). \qquad \square$$

3.2.4 Exercise 4

For the linear operator T of Exercise 3.2.3, prove that

$$(T^2 - I)(T - 3I) = 0.$$

Proof. We have

$$\begin{aligned} T^2(x_1, x_2, x_3) &= T(3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) \\ &= (9x_1, 2x_1 + x_2, 9x_1 + x_3), \end{aligned}$$

so

$$(T^2 - I)(x_1, x_2, x_3) = (8x_1, x_1, 8x_1).$$

Also,

$$(T - 3I)(x_1, x_2, x_3) = (0, x_1 - 4x_2, 2x_1 + x_2 - 2x_3).$$

Consequently,

$$\begin{aligned} (T^2 - I)(T - 3I)(x_1, x_2, x_3) &= (T^2 - I)(0, x_1 - 4x_2, 2x_1 + x_2 - 2x_3) \\ &= (0, 0, 0). \end{aligned}$$

Therefore $(T^2 - I)(T - 3I) = 0$. $\square$

3.2.5 Exercise 5

Let $C^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$$

and let T be the linear operator on $C^{2\times 2}$ defined by $T(A) = BA$. What is the rank of T? Can you describe T^2?

Solution. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a matrix in $C^{2\times 2}$. Then

$$T(A) = BA = \begin{bmatrix} a-c & b-d \\ -4a+4c & -4b+4d \end{bmatrix}.$$

We see that $T(A) = 0$ if and only if both $a = c$ and $b = d$. Consequently, a basis for the null space of T is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Therefore the nullity of T is 2. By Theorem 2, the rank of T is $4 - 2 = 2$.

Since

$$B^2 = \begin{bmatrix} 5 & -5 \\ -20 & 20 \end{bmatrix} = 5 \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} = 5B,$$

we see that $T^2(A) = 5BA = 5T(A)$. □

3.2.6 Exercise 6

Let T be a linear transformation from R^3 into R^2, and let U be a linear transformation from R^2 into R^3. Prove that the transformation UT is not invertible. Generalize the theorem.

Solution. We will state and prove the more general result directly. Let V and W be finite-dimensional vector spaces over the same field F, and suppose that $\dim V > \dim W$. Let T be a linear transformation from V into W and let U be a linear transformation from W into V. Then the transformation UT is not invertible, as we will now show.

First, since the rank of T is at most $\dim W < \dim V$, it follows that the nullity of T is greater than 0. Thus T is not one to one, and there are distinct vectors α and β in V such that $T\alpha = T\beta$. Then we have

$$UT(\alpha - \beta) = U(T(\alpha) - T(\beta)) = U(0) = 0,$$

so UT is not one to one. This shows that UT is not invertible. □

3.2.7 Exercise 7

Find two linear operators T and U on R^2 such that $TU = 0$ but $UT \neq 0$.

Solution. Let T and U be given by

$$T(x_1, x_2) = (x_1, 0) \quad \text{and} \quad U(x_1, x_2) = (0, x_1).$$

Then $TU(x_1, x_2) = T(0, x_1) = (0, 0)$ as required. We also have $UT \neq 0$ since

$$UT(1, 1) = U(1, 0) = (0, 1) \neq (0, 0).$$

□

3.2.8 Exercise 8

Let V be a vector space over the field F and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on R^2 such that $T^2 = 0$ but $T \neq 0$.

Solution. Let β be in the range of T. Then there is an α in V such that $T\alpha = \beta$. But then

$$T\beta = T(T\alpha) = T^2(\alpha) = 0,$$

so β is in the null space of T. This shows that the range of T is contained in the null space of T.

On R^2, define T by

$$T(x_1, x_2) = (x_2, 0).$$

Then

$$T^2(x_1, x_2) = T(x_2, 0) = (0, 0),$$

so $T^2 = 0$ but $T \neq 0$. □

3.2.9 Exercise 9

Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional.

Solution. Let α be in the null space of U, i.e., let $U\alpha = 0$. Then

$$TU(\alpha) = T(U\alpha) = T(0) = 0.$$

Thus α is in the null space of TU. But $TU = I$, so this implies that $\alpha = 0$. This shows that U is non-singular. By Theorem 9, U is invertible.

Since $TU = I$, we have by the associativity of function composition that

$$U^{-1} = (TU)U^{-1} = T(UU^{-1}) = T.$$

But if $T = U^{-1}$, then by definition T is invertible and $U = T^{-1}$.

To show that the original statement is not true when we remove the requirement that V be finite-dimensional, let V be the space of polynomial functions over F, where F has characteristic zero. Let $T = D$, the differentiation operator, and let $U = E$, the integration operator, as defined in Example 11. Then T and U are linear operators on V such that $TU = I$, but T is not invertible since the differentiation operator is singular (its null space consists of all constant functions). □

3.2.10 Exercise 10

Let A be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$ defined by $T(X) = AX$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

Solution. Let $\mathcal{B} = \{\epsilon_1, \ldots, \epsilon_n\}$ be the standard ordered basis for $F^{n\times 1}$ and let $\mathcal{B}' = \{\epsilon_1', \ldots, \epsilon_m'\}$ be the standard ordered basis for $F^{m\times 1}$.

Now suppose $m < n$. Let A be the $m \times n$ matrix whose jth column, for $1 \leq j \leq m$, is ϵ_j', and whose remaining columns are zero. Then the linear transformation $T(X) = AX$ is such that

$$T(\epsilon_j) = \epsilon_j', \quad \text{for each } j \text{ with } 1 \leq j \leq m,$$

and

$$T(\epsilon_j) = 0 \quad \text{for } m < j \leq n.$$

Since every vector in $\mathcal{B}'$ is in the range of T, we see that T is onto. However it is not possible for T to be non-singular, since by Theorem 2 the nullity of T must be $n - m > 0$.

On the other hand, assume $m > n$. Take A to be the $m \times n$ matrix whose ith row is ϵ_i for $1 \leq i \leq n$, with remaining rows zero. Then $AX = 0$ if and only if $X = 0$, so that the nullspace of T is $\{0\}$. But the rank of T is $n < m$, so T is non-singular but not onto. □

3.2.11 Exercise 11

Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Proof. Note that the null space of T is contained in the null space of T^2, since if $T\alpha = 0$ then $T^2(\alpha) = T(0) = 0$. But T and T^2 have the same rank, so by Theorem 2 they must have the same nullity. So any basis for the null space of T must also be a basis for the null space of T^2. It follows that the two null spaces are exactly equal.

Now let β be in the intersection of the range and null space of T. Then there is an α in V with $T\alpha = \beta$. This implies that

$$T^2(\alpha) = T\beta = 0,$$

so α is in the null space of T^2. But T and T^2 have the same null space, so α is in the null space of T. Hence

$$\beta = T\alpha = 0.$$

This shows that the intersection of the range of T and the null space of T is precisely the set $\{0\}$. □

3.2.12 Exercise 12

Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Proof. First assume that $p = m$ (so that $V = W$) and that B is invertible. Define the linear transformation U from V into V by $U(A) = B^{-1}A$. Then for any A in V, we have

$$TU(A) = T(B^{-1}A) = B(B^{-1}A) = (BB^{-1})A = A$$

and

$$UT(A) = U(BA) = B^{-1}(BA) = (B^{-1}B)A = A.$$

This shows that $TU = UT = I$, so by definition T is invertible and $T^{-1} = U$. The first half of the proof is complete.

Next, for the converse, assume that T is invertible. Then T is non-singular, so the nullity of T is 0. By Theorem 2, we have

$$\operatorname{rank}(T) = \dim V = mn.$$

On the other hand, T is onto, so

$$\operatorname{rank}(T) = \dim W = pn.$$

Therefore $mn = pn$ and we see that $p = m$ and $V = W$. Now B is an $m \times m$ matrix. If we define $C = T^{-1}(I)$, then

$$T(CA) = B(CA) = (BC)A = T(C) \cdot A = IA = A,$$

so $T^{-1}(A) = CA$. Since $TT^{-1} = T^{-1}T = I$, it follows that $BC = CB = I$ so that B is invertible. This completes the proof. $\square$

3.3 Isomorphism

3.3.1 Exercise 1

Let V be the set of complex numbers and let F be the field of real numbers. With the usual operations, V is a vector space over F. Describe explicitly an isomorphism of this space onto R^2.

Solution. Define the map T from V into R^2 by

$$T(a+bi) = (a,b), \quad \text{where } a \text{ and } b \text{ belong to } R.$$

Then for any c in R,

$$\begin{aligned} T(c(a_1+b_1i)+(a_2+b_2i)) &= T((ca_1+a_2)+(cb_1+b_2)i) \\ &= (ca_1+a_2, cb_1+b_2) \\ &= c(a_1,b_1)+(a_2,b_2) \\ &= cT(a_1+b_1i)+T(a_2+b_2i), \end{aligned}$$

so T is a linear transformation. It is one to one, since

$$(a,b)=(c,d) \quad \text{implies} \quad a+bi=c+di,$$

and it is onto since (a,b) is evidently in the range of T for all a,b in R. Therefore T is an isomorphism and V and R^2 are isomorphic. □

3.3.2 Exercise 2

Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V onto C^3. Let α_1, α_2, α_3, α_4 be vectors in V such that

$$\begin{gathered} T\alpha_1 = (1,0,i) \quad T\alpha_2 = (-2,1+i,0), \\ T\alpha_3 = (-1,1,1), \quad T\alpha_4 = (\sqrt{2},i,3). \end{gathered}$$

(a) Is α_1 in the subspace spanned by α_2 and α_3?

Solution. If $T\alpha_1$ is in the subspace of C^3 spanned by $T\alpha_2$ and $T\alpha_3$, then there is x_1, x_2 in C with

$$\begin{aligned} -2x_1 - x_2 &= 1 \\ (1+i)x_1 + x_2 &= 0 \\ x_2 &= i. \end{aligned}$$

We reduce the augmented matrix for this system of equations to get

$$\begin{bmatrix} -2 & -1 & 1 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix},$$

so

$$T\alpha_1 = \left(-\frac{1}{2}-\frac{1}{2}i\right)T\alpha_2 + iT\alpha_3.$$

Since T is an isomorphism, it follows that

$$T\alpha_1 = T\left(\left(-\frac{1}{2}-\frac{1}{2}i\right)\alpha_2 + i\alpha_3\right)$$

or

$$\alpha_1 = \left(-\frac{1}{2}-\frac{1}{2}i\right)\alpha_2 + i\alpha_3.$$

Therefore α_1 is in the subspace spanned by α_2 and α_3. $\square$

(b) Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2?

Solution. Let α be in $W_1 \cap W_2$. Then α is a linear combination of α_1 and α_2, and also a linear combination of α_3 and α_4. We can therefore find c_1, c_2, c_3, and c_4 in C with

$$\alpha = c_1\alpha_1 + c_2\alpha_2 = c_3\alpha_3 + c_4\alpha_4.$$

So

$$T(c_1\alpha_1 + c_2\alpha_2 - c_3\alpha_3 - c_4\alpha_4) = 0$$

which implies that

$$c_1T\alpha_1 + c_2T\alpha_2 - c_3T\alpha_3 - c_4T\alpha_4 = 0.$$

This then leads to the system of equations

$$\begin{aligned} c_1 - \quad 2c_2 + c_3 - \sqrt{2}c_4 &= 0 \\ (1+i)c_2 - c_3 - \quad ic_4 &= 0 \\ ic_1 \quad\quad - c_3 - \quad 3c_4 &= 0. \end{aligned}$$

The coefficient matrix for this system reduces to

$$\begin{bmatrix} 1 & -2 & 1 & -\sqrt{2} \\ 0 & 1+i & -1 & -i \\ i & 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, letting $t = -2c_3$, we get

$$2it\alpha_1 + (i-1)t\alpha_2 + 2t\alpha_3 = 0.$$

In particular, $c_4 = 0$ and we see that $\alpha = -2t\alpha_3$ where t is arbitrary. The space $W_1 \cap W_2$ therefore has $\{\alpha_3\}$ as a basis. Consequently, this space is the one-dimensional subspace consisting of scalar multiples of α_3. $\square$

(c) Find a basis for the subspace of V spanned by the four vectors α_j.

Solution. We have already seen previously that α_3 can be written as a linear combination of α_1 and α_2:

$$\alpha_3 = -i\alpha_1 + \frac{1-i}{2}\alpha_2.$$

A check will show that the remaining vectors are linearly independent. Therefore $\{\alpha_1, \alpha_2, \alpha_4\}$ forms a basis for the subspace of V spanned by α_j. $\square$

3.3.3 Exercise 3

Let W be the set of all 2×2 complex Hermitian matrices, that is, the set of 2×2 complex matrices A such that $A_{ij} = \overline{A_{ji}}$ (the bar denoting complex conjugation). As we pointed out in Example 6 of Chapter 2, W is a vector space over the field of *real* numbers, under the usual operations. Verify that

$$(x, y, z, t) \to \begin{bmatrix} t+x & y+iz \\ y-iz & t-x \end{bmatrix}$$

is an isomorphism of R^4 onto W.

Proof. Denote this mapping by T. Then for any

$$\alpha = (x_1, y_1, z_1, t_1) \quad \text{and} \quad \beta = (x_2, y_2, z_2, t_2)$$

in R^4 and any c in R, we have

$$\begin{aligned} T(c\alpha + \beta) &= T(cx_1 + x_2, cy_1 + y_2, cz_1 + z_2, ct_1 + t_2) \\ &= \begin{bmatrix} (ct_1 + t_2) + (cx_1 + x_2) & (cy_1 + y_2) + i(cz_1 + z_2) \\ (cy_1 + y_2) - i(cz_1 + z_2) & (ct_1 + t_2) - (cx_1 + x_2) \end{bmatrix} \\ &= c \begin{bmatrix} t_1 + x_1 & y_1 + iz_1 \\ y_1 - iz_1 & t_1 - x_1 \end{bmatrix} + \begin{bmatrix} t_2 + x_2 & y_2 + iz_2 \\ y_2 - iz_2 & t_2 - x_2 \end{bmatrix} \\ &= cT\alpha + T\beta. \end{aligned}$$

This shows that T is a linear transformation.

Next, if $T\alpha = T\beta$ then $t_1 + x_1 = t_2 + x_2$ and $t_1 - x_1 = t_2 - x_2$, which together imply that $t_1 = t_2$ and $x_1 = x_2$. Similarly, $y_1 + iz_1 = y_2 + iz_2$ implies that $y_1 = y_2$ and $z_1 = z_2$. Therefore T is one to one.

Finally, let

$$A = \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix}$$

be any 2×2 Hermitian matrix. Then we see that

$$T\left(\frac{1}{2}a + \frac{1}{2}d, b, c, \frac{1}{2}a - \frac{1}{2}d\right) = \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix} = A,$$

so T is onto. This shows that T is an isomorphism and R^4 is isomorphic to W. □

3.3.4 Exercise 4

Show that $F^{m\times n}$ is isomorphic to F^{mn}.

Proof. An obvious isomorphism is the map T from $F^{m\times n}$ onto F^{mn} given by

$$\begin{aligned} &T \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \\ &\qquad = (A_{11}, A_{12}, \ldots, A_{1n}, A_{21}, A_{22}, \ldots, A_{2n}, \ldots, A_{m1}, A_{m2}, \ldots, A_{mn}). \end{aligned}$$

That is, the jth coordinate of $T(A)$ is the jth entry of A when the entries are ordered from left-to-right and then top-to-bottom. It should be evident that T is a linear transformation that is both one to one and onto. □

3.3.5 Exercise 5

Let V be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from V into the space of 2×2 real matrices, as follows. If $z = x + iy$ with x and y real numbers, then

$$T(z) = \begin{bmatrix} x + 7y & 5y \\ -10y & x - 7y \end{bmatrix}.$$

(a) Verify that T is a one-one (real) linear transformation of V into the space of 2×2 real matrices.

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where x_1, x_2, y_1, y_2 are real numbers. Let c be any real number. Then

$$\begin{aligned} T(cz_1 + z_2) &= T(cx_1 + x_2 + i(cy_1 + y_2)) \\ &= \begin{bmatrix} (cx_1 + x_2) + 7(cy_1 + y_2) & 5(cy_1 + y_2) \\ -10(cy_1 + y_2) & (cx_1 + x_2) - 7(cy_1 + y_2) \end{bmatrix} \\ &= c \begin{bmatrix} x_1 + 7y_1 & 5y_1 \\ -10y_1 & x_1 - 7y_1 \end{bmatrix} + \begin{bmatrix} x_2 + 7y_2 & 5y_2 \\ -10y_2 & x_2 - 7y_2 \end{bmatrix} \\ &= cT(z_1) + T(z_2) \end{aligned}$$

and we see that T is a linear transformation.

Moreover, if $T(z_1) = T(z_2)$ then $5y_1 = 5y_2$ so that $y_1 = y_2$ and $x_1 = x_2$, or $z_1 = z_2$. Thus T is one to one. □

(b) Verify that $T(z_1 z_2) = T(z_1)T(z_2)$.

Proof. As above, we let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$T(z_1)T(z_2) = \begin{bmatrix} x_1 + 7y_1 & 5y_1 \\ -10y_1 & x_1 - 7y_1 \end{bmatrix} \begin{bmatrix} x_2 + 7y_2 & 5y_2 \\ -10y_2 & x_2 - 7y_2 \end{bmatrix}.$$

If we calculate the upper-left entry of this matrix product, we find

$$\begin{aligned} [T(z_1)T(z_2)]_{11} &= (x_1 + 7y_1)(x_2 + 7y_2) - 50y_1y_2 \\ &= x_1x_2 + 7x_1y_2 + 7x_2y_1 + 49y_1y_2 - 50y_1y_2 \\ &= (x_1x_2 - y_1y_2) + 7(x_1y_2 + x_2y_1) \\ &= [T(z_1z_2)]_{11}. \end{aligned}$$

The remaining entries are calculated in the same manner, and are all straightforward. Therefore $T(z_1z_2) = T(z_1)T(z_2)$. □

(c) How would you describe the range of T?

Solution. We have shown above that the range of T is isomorphic to V. A basis for V is $\{1, i\}$, so we may compute

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T(i) = \begin{bmatrix} 7 & 5 \\ -10 & -7 \end{bmatrix}.$$

Since T preserves linear independence (Theorem 8), we see that the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 5 \\ -10 & -7 \end{bmatrix} \right\}$$

is a basis for the range of T. □

3.3.6 Exercise 6

Let V and W be finite-dimensional vector spaces over the field F. Prove that V and W are isomorphic if and only if $\dim V = \dim W$.

Proof. First suppose that V and W are isomorphic via the isomorphism T. Suppose V has dimension n and let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Then by Theorem 8, the set $\{T\alpha_1, \ldots, T\alpha_n\}$ is linearly independent and thus forms a basis for the range of T. But T is onto, so the range of T is W. Therefore $\dim W = n$ as required.

Conversely, suppose $\dim V = \dim W = n$. By Theorem 10, V and W are both isomorphic to F^n. Thus V is isomorphic to W and the proof is complete. □

3.3.7 Exercise 7

Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V,V)$ onto $L(W,W)$.

Proof. Let S denote the stated map from $L(V,V)$ to $L(W,W)$. If c is in F, then

$$\begin{aligned} S(cT_1 + T_2) &= U(cT_1 + T_2)U^{-1} \\ &= (cUT_1 + UT_2)U^{-1} \\ &= cUT_1U^{-1} + UT_2U^{-1} \\ &= cS(T_1) + S(T_2), \end{aligned}$$

so S is a linear transformation.

Next, suppose $S(T_1) = S(T_2)$. That is, $UT_1U^{-1} = UT_2U^{-1}$. Then

$$\begin{aligned} T_1 &= (U^{-1}U)T_1(U^{-1}U) \\ &= U^{-1}(UT_1U^{-1})U \\ &= U^{-1}(UT_2U^{-1})U \\ &= (U^{-1}U)T_2(U^{-1}U) \\ &= T_2, \end{aligned}$$

showing that S is one to one.

Finally, let T be any linear operator in $L(W,W)$. Then

$$S(U^{-1}TU) = U(U^{-1}TU)U^{-1} = (UU^{-1})T(UU^{-1}) = T,$$

so S is onto. This shows that S is an isomorphism, so that $L(V,V)$ is isomorphic to $L(W,W)$. □

3.4 Representation of Transformations by Matrices

3.4.1 Exercise 1

Let T be the linear operator on C^2 defined by $T(x_1, x_2) = (x_1, 0)$. Let $\mathcal{B}$ be the standard ordered basis for C^2 and let $\mathcal{B}' = \{\alpha_1, \alpha_2\}$ be the ordered basis defined by $\alpha_1 = (1, i)$, $\alpha_2 = (-i, 2)$.

(a) What is the matrix of T relative to the pair $\mathcal{B}$, $\mathcal{B}'$?

Solution. We have

$$T(1,0) = (1,0) \quad \text{and} \quad T(0,1) = (0,0).$$

Now let

$$P = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}$$

and we get

$$[(1,0)]_{\mathcal{B}'} = P^{-1}[(1,0)]_{\mathcal{B}} = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -i \end{bmatrix}.$$

Of course, the zero vector has the same coordinates in every basis, so we see that the matrix of T relative to $\mathcal{B}, \mathcal{B}'$ is

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = \begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}. \qquad \square$$

(b) What is the matrix of T relative to the pair $\mathcal{B}'$, $\mathcal{B}$?

Solution. We have

$$T(1,i) = (1,0) \quad \text{and} \quad T(-i,2) = (-i,0).$$

So the matrix of T relative to $\mathcal{B}', \mathcal{B}$ is

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}. \qquad \square$$

(c) What is the matrix of T in the ordered basis $\mathcal{B}'$?

Solution. We have

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2i \\ -i & -1 \end{bmatrix}. \qquad \square$$

(d) What is the matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$?

Solution. The change-of-basis matrix P such that

$$P[\alpha]_{\{\alpha_2,\alpha_1\}} = [\alpha]_{\mathcal{B}'}$$

is given by

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so

$$[T]_{\{\alpha_2,\alpha_1\}} = P^{-1}[T]_{\mathcal{B}'}P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2i \\ -i & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ -2i & 2 \end{bmatrix}. \quad \square$$

3.4.2 Exercise 2

Let T be the linear transformation from R^3 into R^2 defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

(a) If $\mathcal{B}$ is the standard ordered basis for R^3 and $\mathcal{B}'$ is the standard ordered basis for R^2, what is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

Solution. Since

$$T(1,0,0) = (1,-1), \quad T(0,1,0) = (1,0), \quad \text{and} \quad T(0,0,1) = (0,2),$$

we see that the matrix of T relative to $\mathcal{B}, \mathcal{B}'$ is

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}. \quad \square$$

(b) If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{B}' = \{\beta_1, \beta_2\}$, where

$$\alpha_1 = (1,0,-1), \quad \alpha_2 = (1,1,1), \quad \alpha_3 = (1,0,0),$$
$$\beta_1 = (0,1), \quad \beta_2 = (1,0)$$

what is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

Solution. We have

$$\begin{aligned} T\alpha_1 &= (1,-3) = -3\beta_1 + \beta_2, \\ T\alpha_2 &= (2,1) = \beta_1 + 2\beta_2, \end{aligned}$$

and

$$T\alpha_3 = (1,-1) = -\beta_1 + \beta_2,$$

so the corresponding matrix is

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}. \quad \square$$

3.4.3 Exercise 3

Let T be a linear operator on F^n, let A be the matrix of T in the standard ordered basis for F^n, and let W be the subspace of F^n spanned by the column vectors of A. What does W have to do with T?

Solution. W is simply the range of T, as we will now show.

Let $\{\epsilon_1, \ldots, \epsilon_n\}$ denote the standard ordered basis for F^n. Note that the jth column of A is simply $T\epsilon_j$. Take any vector α in F^n. Then α belongs to W if and only if

$$\begin{aligned}\alpha &= x_1 T\epsilon_1 + x_2 T\epsilon_2 + \cdots + x_n T\epsilon_n \\ &= T(x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n) \\ &= T(x_1, x_2, \cdots, x_n),\end{aligned}$$

for some vector $(x_1, x_2, \ldots, x_n)$ in F^n. That is, α is in W if and only if α is in the range of T. □

3.4.4 Exercise 4

Let V be a two-dimensional vector space over the field F, and let $\mathcal{B}$ be an ordered basis for V. If T is a linear operator on V and

$$[T]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

prove that $T^2 - (a+d)T + (ad - bc)I = 0$.

Proof. By Theorem 12, the function which assigns a linear operator on V to its matrix relative to $\mathcal{B}$ is an isomorphism between $L(V, V)$ and $F^{2\times 2}$. Theorem 13 shows that this function preserves products also. Thus we can operate on T by simply performing the corresponding operations on its matrix and vice versa. So consider the following computation.

$$\begin{aligned}&[T]_{\mathcal{B}}^2 - (a+d)[T]_{\mathcal{B}} + (ad-bc)I \\ &\qquad = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

From this we see that $T^2 - (a+d)T + (ad-bc)I = 0$. □

3.4.5 Exercise 5

Let T be the linear operator on R^3, the matrix of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find a basis for the range of T and a basis for the null space of T.

Solution. By Exercise 3.4.3, we know that the column space of A is the range of T. We can find a basis for the column space of A by row-reducing its transpose A^T and taking the nonzero rows. We get

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix},$$

so a basis for the range of T is given by

$$\{(1, 0, -1), (0, 1, 5)\}.$$

For the nullspace, we seek column vectors X for which $AX = 0$. Row-reducing A gives

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so (x_1, x_2, x_3) is in the null space of T if and only if $x_1 = x_3$ and $x_2 = -x_3$. That is, if and only if (x_1, x_2, x_3) has the form $(t, -t, t)$ for some scalar t. A basis for the null space of T is therefore given by

$$\{(1, -1, 1)\}. \qquad \square$$

3.4.6 Exercise 6

Let T be the linear operator on R^2 defined by

$$T(x_1, x_2) = (-x_2, x_1).$$

(a) What is the matrix of T in the standard ordered basis for R^2?

Solution. Since

$$T(1, 0) = (0, 1) \quad \text{and} \quad T(0, 1) = (-1, 0),$$

the matrix of T relative to the standard ordered basis is

$$[T]_{\{\epsilon_1, \epsilon_2\}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \qquad \square$$

(b) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where

$$\alpha_1 = (1, 2) \quad \text{and} \quad \alpha_2 = (1, -1)?$$

Solution. The transition matrix P from $\mathcal{B}$ to the standard basis is

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \text{with inverse} \quad P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

So,

$$[T]_{\mathcal{B}} = P^{-1}[T]_{\{\epsilon_1, \epsilon_2\}} P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{bmatrix}. \qquad \square$$

(c) Prove that for every real number c the operator $(T - cI)$ is invertible.

Proof. Fix c in R and let $U = T - cI$. If $\mathcal{B}$ is the standard ordered basis for R^2, then

$$[U]_{\mathcal{B}} = [T]_{\mathcal{B}} - cI = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -c & -1 \\ 1 & -c \end{bmatrix}.$$

Since $c^2 + 1$ must be nonzero, it is easily verified that

$$\begin{bmatrix} -\frac{c}{c^2+1} & \frac{1}{c^2+1} \\ -\frac{1}{c^2+1} & -\frac{c}{c^2+1} \end{bmatrix} = -\frac{1}{c^2+1}\begin{bmatrix} c & -1 \\ 1 & c \end{bmatrix}$$

is an inverse for $[U]_{\mathcal{B}}$. It follows that U is invertible, as required. □

(d) Prove that if $\mathcal{B}$ is any ordered basis for R^2 and $[T]_{\mathcal{B}} = A$, then $A_{12}A_{21} \neq 0$.

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2\}$ where

$$\alpha_1 = (a, c) \quad \text{and} \quad \alpha_2 = (b, d)$$

so that

$$[T]_{\mathcal{B}} = P^{-1}[T]_{\{\epsilon_1, \epsilon_2\}}P,$$

where

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since P is invertible, we know from Exercise 1.6.8 that $ad - bc \neq 0$, and through direct computation we can find that

$$A = [T]_{\mathcal{B}} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{-cd-ab}{ad-bc} & \frac{-d^2-b^2}{ad-bc} \\ \frac{c^2+a^2}{ad-bc} & \frac{cd+ab}{ad-bc} \end{bmatrix}.$$

Now, if $A_{12} = 0$, then $b^2 + d^2 = 0$ so that we must have $b = d = 0$. But this is impossible since $ad - bc \neq 0$. Similarly if $A_{21} = 0$ then $a = c = 0$ which is again a contradiction. So we find that A_{12} and A_{21} are each nonzero, and their product must be nonzero also. □

3.4.7 Exercise 7

Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

(a) What is the matrix of T in the standard ordered basis for R^3?

Solution. Since

$$\begin{aligned} T(1,0,0) &= (3,-2,-1), \\ T(0,1,0) &= (0,1,2), \\ T(0,0,1) &= (1,0,4), \end{aligned}$$

the matrix of T in the standard ordered basis is

$$[T]_{\{\epsilon_1,\epsilon_2,\epsilon_3\}} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}.$$ □

(b) What is the matrix of T in the ordered basis

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$, and $\alpha_3 = (2, 1, 1)$?

Solution. The matrix of T in this basis is

$$[T]_{\{\alpha_1,\alpha_2,\alpha_3\}} = P^{-1}[T]_{\{\epsilon_1,\epsilon_2,\epsilon_3\}}P,$$

where

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} [T]_{\{\alpha_1,\alpha_2,\alpha_3\}} &= \begin{bmatrix} -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{17}{4} & \frac{35}{4} & \frac{11}{2} \\ -\frac{3}{4} & \frac{15}{4} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{7}{2} & 0 \end{bmatrix}. \end{aligned}$$

□

(c) Prove that T is invertible and give a rule for T^{-1} like the one which defines T.

Proof. We already saw that the matrix A of T relative to the standard ordered basis of R^3 is

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}.$$

Using the same methods as we used in Chapter 1, we can see that A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & -\frac{1}{9} \\ \frac{8}{9} & \frac{13}{9} & -\frac{2}{9} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

This implies that T is invertible and

$$\begin{aligned} &T^{-1}(x_1, x_2, x_3) \\ &\quad = \left(\frac{4}{9}x_1 + \frac{2}{9}x_2 - \frac{1}{9}x_3, \frac{8}{9}x_1 + \frac{13}{9}x_2 - \frac{2}{9}x_3, -\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3\right). \end{aligned}$$

□

3.4.8 Exercise 8

Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Proof. Let T be the linear operator on C^2 which is represented by the first matrix in the standard ordered basis. Let

$$\alpha_1 = (1, -i) \quad \text{and} \quad \alpha_2 = (-i, 1),$$

so that $\mathcal{B} = \{\alpha_1, \alpha_2\}$ is an ordered basis for C^2. Then

$$T\alpha_1 = (\cos\theta + i\sin\theta, \sin\theta - i\cos\theta) = (e^{i\theta}, -ie^{i\theta}) = e^{i\theta}\alpha_1$$

and

$$T\alpha_2 = (-i\cos\theta - \sin\theta, -i\sin\theta + \cos\theta) = (-ie^{-i\theta}, e^{-i\theta}) = e^{-i\theta}\alpha_2.$$

Thus the second matrix represents T in the ordered basis $\mathcal{B}$. By Theorem 14, the two matrices are similar. □

3.4.9 Exercise 9

Let V be a finite-dimensional vector space over the field F and let S and T be linear operators on V. We ask: When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for V such that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$? Prove that such bases exist if and only if there is an invertible linear operator U on V such that $T = USU^{-1}$.

Proof. Assume such bases exist, so that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$. Let U be the operator which carries $\mathcal{B}$ onto $\mathcal{B}'$. Then by Theorem 14, we have

$$[S]_{\mathcal{B}} = [T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}} = [U^{-1}TU]_{\mathcal{B}}.$$

Since S and $U^{-1}TU$ have the same matrix relative to $\mathcal{B}$, it follows by Theorem 12 that $S = U^{-1}TU$. Thus we have shown that there is an invertible operator U with $T = USU^{-1}$.

Conversely, assume that $T = USU^{-1}$ for some invertible U and let $\mathcal{B}$ be any ordered basis for V. Let $\mathcal{B}'$ be the image of $\mathcal{B}$ under U. Then, again by Theorem 14, we have

$$[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}} = [U^{-1}TU]_{\mathcal{B}} = [S]_{\mathcal{B}}.$$

Thus the proof is complete. □

3.4.10 Exercise 10

We have seen that the linear operator T on R^2 defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This operator satisfies $T^2 = T$. Prove that if S is a linear operator on R^2 such that $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis $\mathcal{B}$ for R^2 such that $[S]_{\mathcal{B}} = A$ (above).

Proof. Assume that S is such that $S^2 = S$. Certainly $0^2 = 0$ and $I^2 = I$, so if $S = 0$ or $S = I$ there is nothing left to prove. We will therefore assume that $S \neq 0$ and $S \neq I$. Since $S \neq 0$, there is nonzero α_1 in the range of S. So $S\beta_1 = \alpha_1$ for some β_1 in R^2 and we have

$$S\alpha_1 = S(S\beta_1) = S\beta_1 = \alpha_1. \tag{3.1}$$

On the other hand, since $S \neq I$, there exist distinct β_2 and β_3 such that

$$S\beta_2 = \beta_3.$$

Applying S to both sides, we get $S^2\beta_2 = S\beta_3$ or $S\beta_2 = S\beta_3$. Now letting $\alpha_2 = \beta_3 - \beta_2$, we get

$$S\alpha_2 = S\beta_3 - S\beta_2 = 0. \tag{3.2}$$

Next, if $c_1\alpha_1 + c_2\alpha_2 = 0$ for scalars c_1 and c_2, then

$$c_1 S\alpha_1 + c_2 S\alpha_2 = 0.$$

Substituting equations (3.1) and (3.2), we see $c_1\alpha_1 = 0$, which implies that $c_1 = 0$ since α_1 is nonzero. So $c_2\alpha_2 = 0$ and we must have $c_2 = 0$ since α_2 is also nonzero. Thus the set $\mathcal{B} = \{\alpha_1, \alpha_2\}$ is a set of two linearly independent vectors in R^2. Therefore $\mathcal{B}$ spans R^2 and is a basis.

Finally, we see that (3.1) and (3.2) together imply that

$$[S]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A,$$

so the proof is complete. □

3.4.11 Exercise 11

Let W be the space of all $n \times 1$ column matrices over a field F. If A is an $n \times n$ matrix over F, then A defines a linear operator L_A on W through left multiplication: $L_A(X) = AX$. Prove that every linear operator on W is left multiplication by some $n \times n$ matrix, i.e., is L_A for some A.

Now suppose V is an n-dimensional vector space over the field F, and let $\mathcal{B}$ be an ordered basis for V. For each α in V, define $U\alpha = [\alpha]_{\mathcal{B}}$. Prove that U is an isomorphism of V onto W. If T is a linear operator on V, then UTU^{-1} is a linear operator on W. Accordingly, UTU^{-1} is left multiplication by some $n \times n$ matrix A. What is A?

Solution. Let $\mathcal{C}$ be the standard ordered basis for $W = F^{n\times 1}$, i.e. the jth vector in the basis has a 1 in its jth row and all other entries zero. Then if S is any linear operator on W, we may take the $n \times n$ matrix A to be

$$A = [S]_{\mathcal{C}}.$$

Then L_A and S have the same matrix relative to the ordered basis $\mathcal{C}$, and therefore $L_A = S$. Therefore every linear operator on W is left multiplication by some $n \times n$ matrix.

Now let V be an n-dimensional vector space over F and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ an ordered basis for V. Define $U\alpha = [\alpha]_{\mathcal{B}}$, so that U is a map of V into W. We will show that U is an isomorphism.

First, for any β_1, β_2 in V and scalar c in F, we have

$$\begin{aligned} U(c\beta_1 + \beta_2) &= [c\beta_1 + \beta_2]_{\mathcal{B}} \\ &= c[\beta_1]_{\mathcal{B}} + [\beta_2]_{\mathcal{B}} \\ &= cU\beta_1 + U\beta_2, \end{aligned}$$

and U is a linear transformation.

Now choose Y in W, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Define

$$\beta = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n.$$

Then

$$U\beta = [\beta]_{\mathcal{B}} = Y.$$

So Y is in the range of U, and U is therefore onto. By Theorem 9, U is invertible and therefore an isomorphism of V onto W.

Finally, if T is a linear operator on V, then UTU^{-1} is a linear operator on W and hence $UTU^{-1} = L_A$ for some $n \times n$ matrix A. Choose any X in W, with

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then

$$\begin{aligned} UTU^{-1}(X) &= UT(x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n) \\ &= U(x_1T\alpha_1 + x_2T\alpha_2 + \cdots + x_nT\alpha_n) \\ &= x_1[T\alpha_1]_{\mathcal{B}} + x_2[T\alpha_2]_{\mathcal{B}} + \cdots + x_n[T\alpha_n]_{\mathcal{B}} \\ &= x_1[T]_{\mathcal{B}}[\alpha_1]_{\mathcal{B}} + x_2[T]_{\mathcal{B}}[\alpha_2]_{\mathcal{B}} + \cdots + x_n[T]_{\mathcal{B}}[\alpha_n]_{\mathcal{B}} \\ &= [T]_{\mathcal{B}}(x_1U\alpha_1 + x_2U\alpha_2 + \cdots + x_nU\alpha_n) \\ &= [T]_{\mathcal{B}}X. \end{aligned}$$

We see that $A = [T]_{\mathcal{B}}$, so $UTU^{-1} = L_{[T]_{\mathcal{B}}}$. $\square$

3.4.12 Exercise 12

Let V be an n-dimensional vector space over the field F, and let

$$\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$$

be an ordered basis for V.

(a) According to Theorem 1, there is a unique linear operator T on V such that

$$T\alpha_j = \alpha_{j+1}, \quad j = 1, \dots, n-1, \quad T\alpha_n = 0.$$

What is the matrix A of T in the ordered basis $\mathcal{B}$?

Solution. For each $j = 1, \dots, n-1$, we have $A_{j+1,j} = 1$ and all other entries 0. That is,

$$A = [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

□

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

Proof. Since $T^{n-1}\alpha_1 = \alpha_n \neq 0$, we see that $T^{n-1} \neq 0$. On the other hand,

$$T^n\alpha_1 = T(T^{n-1}\alpha_1) = T\alpha_n = 0,$$

and we can similarly see that $T^n\alpha_j = 0$ for all j with $1 \leq j \leq n$. Since any vector α can be written as a linear combination of the basis vectors $\alpha_1, \dots, \alpha_n$, it follows by the linearity of T that $T^n\alpha = 0$. Therefore $T^n = 0$. □

(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix of S in the ordered basis $\mathcal{B}'$ is the matrix A of part (a).

Proof. Since $S^{n-1} \neq 0$, there is a nonzero β_1 in V such that $S^{n-1}\beta_1$ is nonzero. Now, for each j with $2 \leq j \leq n$, define

$$\beta_j = S^{j-1}\beta_1.$$

Let $\mathcal{B}' = \{\beta_1, \dots, \beta_n\}$.

Now suppose $c_1, c_2, \dots, c_n$ are scalars in F such that

$$c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n = 0.$$

Taking S^{n-1} of both sides gives $c_1\beta_n = 0$, which implies that $c_1 = 0$. So

$$c_2\beta_2 + c_3\beta_3 + \cdots + c_n\beta_n = 0,$$

and we can take S^{n-2} of both sides to get $c_2\beta_n = 0$, implying that c_2 is zero. More generally, assuming that $c_1, \ldots, c_k$ are all zero for some k with $1 \le k \le n-1$, we have

$$c_{k+1}\beta_{k+1} + \cdots + c_n\beta_n = 0.$$

Taking S^{n-k-1} of both sides (in the case where $k = n-1$, we take $S^0 = I$) then gives $c_{k+1}\beta_n = 0$, so that $c_{k+1} = 0$. Therefore $c_1 = c_2 = \cdots = c_n = 0$ and $\mathcal{B}'$ is linearly independent. Since $\dim V = n$ and $\mathcal{B}'$ is a linearly independent set of n vectors in V, it follows that $\mathcal{B}'$ is a basis for V.

Finally, we have defined $\beta_1, \ldots, \beta_n$ so that

$$S\beta_j = \beta_{j+1}, \quad j = 1, \ldots, n-1, \quad S\beta_n = 0.$$

Therefore we have $[S]_{\mathcal{B}'} = A$. □

(d) Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

Proof. Let P and Q be the linear operators on V whose matrices relative to $\mathcal{B}$ are, respectively, M and N. Then $P^n = Q^n = 0$ but $P^{n-1} \neq 0 \neq Q^{n-1}$. We have shown above that there are bases $\mathcal{C}$ and $\mathcal{C}'$ for V such that

$$[P]_{\mathcal{C}} = A = [Q]_{\mathcal{C}'}.$$

By Exercise 3.4.9, there is an invertible linear transformation U such that $Q = UPU^{-1}$. Therefore

$$N = [Q]_{\mathcal{B}} = [U]_{\mathcal{B}}[P]_{\mathcal{B}}[U]_{\mathcal{B}}^{-1} = [U]_{\mathcal{B}}M[U]_{\mathcal{B}}^{-1}$$

and we see that M and N are similar matrices. □

3.4.13 Exercise 13

Let V and W be finite-dimensional vector spaces over the field F and let T be a linear transformation from V into W. If

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad \mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$$

are ordered bases for V and W, respectively, define the linear transformations $E^{p,q}$ as in the proof of Theorem 5: $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$. Then the $E^{p,q}$, $1 \le p \le m$, $1 \le q \le n$, form a basis for $L(V, W)$, and so

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}$$

for certain scalars A_{pq} (the coordinates of T in this basis for $L(V, W)$). Show that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

Solution. Note that for each $j = 1, \dots, n$,

$$\begin{aligned} T\alpha_j &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}(\alpha_j) \\ &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\delta_{jq}\beta_p \\ &= \sum_{p=1}^{m} A_{pj}\beta_p. \end{aligned}$$

Thus the ith coordinate of $T\alpha_j$ is A_{ij} and we see that the matrix of T relative to $\mathcal{B}, \mathcal{B}'$ is precisely the matrix A. $\square$

3.5 Linear Functionals

3.5.1 Exercise 1

In R^3, let $\alpha_1 = (1,0,1)$, $\alpha_2 = (0,1,-2)$, $\alpha_3 = (-1,-1,0)$.

(a) If f is a linear functional on R^3 such that

$$f(\alpha_1) = 1, \quad f(\alpha_2) = -1, \quad f(\alpha_3) = 3,$$

and if $\alpha = (a,b,c)$, find $f(\alpha)$.

Solution. Suppose $f(x_1,x_2,x_3) = c_1x_1 + c_2x_2 + c_3x_3$. Then

$$\begin{aligned} f(\alpha_1) &= c_1 \phantom{{}-c_2} + c_3 = 1, \\ f(\alpha_2) &= \phantom{-c_1-{}} c_2 - 2c_3 = -1, \\ f(\alpha_3) &= -c_1 - c_2 \phantom{{}-2c_3} = 3. \end{aligned}$$

Row-reducing the augmented matrix for this system gives

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ -1 & -1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -3 \end{bmatrix},$$

so $c_1 = 4$, $c_2 = -7$, and $c_3 = -3$. Therefore

$$f(\alpha) = 4a - 7b - 3c.$$

□

(b) Describe explicitly a linear functional f on R^3 such that

$$f(\alpha_1) = f(\alpha_2) = 0 \quad \text{but} \quad f(\alpha_3) \neq 0.$$

Solution. For example, suppose $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) = 1$. As above, this leads to a system of linear equations having augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

So we may write

$$f(x_1,x_2,x_3) = x_1 - 2x_2 - x_3.$$

□

(c) Let f be any linear functional such that

$$f(\alpha_1) = f(\alpha_2) = 0 \quad \text{and} \quad f(\alpha_3) \neq 0.$$

If $\alpha = (2,3,-1)$, show that $f(\alpha) \neq 0$.

Solution. By inspection, we see that

$$\alpha = -\alpha_1 - 3\alpha_3.$$

Therefore

$$\begin{aligned} f(\alpha) &= f(-\alpha_1 - 3\alpha_3) \\ &= -f(\alpha_1) - 3f(\alpha_3) \\ &= -3f(\alpha_3) \neq 0. \end{aligned}$$

□

3.5.2 Exercise 2

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for C^3 defined by

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (2, 2, 0).$$

Find the dual basis of $\mathcal{B}$.

Solution. Let $\{f_1, f_2, f_3\}$ be the dual basis of $\mathcal{B}$, and let

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix},$$

so that P is the transition matrix from $\mathcal{B}$ to the standard ordered basis of C^3. We find

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}.$$

So, given a vector $\alpha = (x_1, x_2, x_3)$ in C_3, we can write

$$[\alpha]_{\mathcal{B}} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore

$$f_1(x_1, x_2, x_3) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 - x_2.$$

Similarly, we get

$$f_2(x_1, x_2, x_3) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 - x_2 + x_3,$$

and

$$f_3(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3. \quad \square$$

3.5.3 Exercise 3

If A and B are $n \times n$ matrices over the field F, show that $\text{trace}(AB) = \text{trace}(BA)$. Now show that similar matrices have the same trace.

Proof. We may directly compute

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ij} \\
&= \sum_{j=1}^{n} (BA)_{jj} \\
&= \operatorname{tr}(BA).
\end{aligned}$$

So $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Next, suppose A and B are similar, and let P be an invertible $n \times n$ matrix such that $B = P^{-1}AP$. Using the fact that was proven above, we get

$$\begin{aligned}
\operatorname{tr}(B) &= \operatorname{tr}(P^{-1}AP) \\
&= \operatorname{tr}((P^{-1}A)P) \\
&= \operatorname{tr}(P(P^{-1}A)) \\
&= \operatorname{tr}((PP^{-1})A) \\
&= \operatorname{tr}(A).
\end{aligned}$$

This shows that similar matrices have the same trace. □

3.5.4 Exercise 4

Let V be the vector space of all polynomial functions p from R into R which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\,dx, \quad f_2(p) = \int_0^2 p(x)\,dx, \quad f_3(p) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.

Solution. First we evaluate,

$$\begin{aligned}
f_1(p) &= \left(c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right) \Bigg|_0^1 = c_0 + \frac{1}{2} c_1 + \frac{1}{3} c_2, \\
f_2(p) &= \left(c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right) \Bigg|_0^2 = 2c_0 + 2c_1 + \frac{8}{3} c_2, \\
f_3(p) &= \left(c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 \right) \Bigg|_0^{-1} = -c_0 + \frac{1}{2} c_1 - \frac{1}{3} c_2.
\end{aligned}$$

Now, let $\{p_1, p_2, p_3\}$ be the basis for V of which $\{f_1, f_2, f_3\}$ is the dual. To determine p_i, we want to find values for the coefficients c_1, c_2, and c_3 so that

$f_i(p_i) = 1$ and $f_j(p_i) = 0$ for $j \neq i$. This gives three systems of linear equations, having augmented matrices

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 2 & 2 & \frac{8}{3} & 0 \\ -1 & \frac{1}{2} & -\frac{1}{3} & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 0 \\ 2 & 2 & \frac{8}{3} & 1 \\ -1 & \frac{1}{2} & -\frac{1}{3} & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 0 \\ 2 & 2 & \frac{8}{3} & 0 \\ -1 & \frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix}.$$

We can combine these into one augmented matrix and perform row-reduction, which gives

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 2 & 2 & \frac{8}{3} & 0 & 1 & 0 \\ -1 & \frac{1}{2} & -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

So we see that

$$\begin{aligned} p_1(x) &= 1 + x - \frac{3}{2}x^2, \\ p_2(x) &= -\frac{1}{6} + \frac{1}{2}x^2, \\ p_3(x) &= -\frac{1}{3} + x - \frac{1}{2}x^2, \end{aligned}$$

and $\{f_1, f_2, f_3\}$ is the dual basis of $\{p_1, p_2, p_3\}$. □

3.5.5 Exercise 5

If A and B are $n \times n$ complex matrices, show that $AB - BA = I$ is impossible.

Proof. In Example 19, the trace function was shown to be a linear functional on the space of $n \times n$ matrices. And in Exercise 3.5.3, we proved that, given two matrices A and B, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. It now follows that

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

But $\operatorname{tr}(I) = n \neq 0$, so it cannot be the case that $AB - BA = I$. □

3.5.6 Exercise 6

Let m and n be positive integers and F a field. Let $f_1, \ldots, f_m$ be linear functionals on F^n. For α in F^n define

$$T\alpha = (f_1(\alpha), \ldots, f_m(\alpha)).$$

Show that T is a linear transformation from F^n into F^m. Then show that every linear transformation from F^n into F^m is of the above form, for some $f_1, \ldots, f_m$.

Proof. First, for any α_1, α_2 in F^n and any c in F, we have

$$\begin{aligned} T(c\alpha_1 + \alpha_2) &= (f_1(c\alpha_1 + \alpha_2), \ldots, f_m(c\alpha_1 + \alpha_2)) \\ &= (cf_1(\alpha_1) + f_1(\alpha_2), \ldots, cf_m(\alpha_1) + f_m(\alpha_2)) \\ &= c(f_1(\alpha_1), \ldots, f_m(\alpha_1)) + (f_1(\alpha_2), \ldots, f_m(\alpha_2)) \\ &= cT\alpha_1 + T\alpha_2. \end{aligned}$$

This shows that T is a linear transformation from F^n into F^m.

Next, let T be any linear transformation from F^n into F^m. For each i with $1 \leq i \leq m$, define $f_i(\alpha)$ to be the ith coordinate of $T\alpha$. Then

$$T\alpha = (f_1(\alpha), \ldots, f_m(\alpha))$$

and, moreover, each f_i is a linear transformation because T itself is linear. Therefore every linear transformation from F^n into F^m can be written this way. $\square$

3.5.7 Exercise 7

Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$, and let W be the subspace of R^4 spanned by α_1 and α_2. Which linear functionals f:

$$f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

are in the annihilator of W?

Solution. Let f be in W^0. We want

$$f(1, 0, -1, 2) = f(2, 3, 1, 1) = 0.$$

This leads to a system of equations in c_1, c_2, c_3, c_4 having coefficient matrix

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

From the reduced form we see that c_3 and c_4 can be arbitrary, with

$$c_1 = c_3 - 2c_4 \quad \text{and} \quad c_2 = c_4 - c_3.$$

Therefore W^0 consists of the linear functionals f having the form

$$f(x_1, x_2, x_3, x_4) = (s - 2t)x_1 + (t - s)x_2 + sx_3 + tx_4,$$

where s and t are scalars in F. Note that we can also find a basis for W^0 by first taking $s = 1, t = 0$ and then by taking $s = 0, t = 1$. $\square$

3.5.8 Exercise 8

Let W be the subspace of R^5 which is spanned by the vectors

$$\alpha_1 = \epsilon_1 + 2\epsilon_2 + \epsilon_3, \quad \alpha_2 = \epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + \epsilon_5$$
$$\alpha_3 = \epsilon_1 + 4\epsilon_2 + 6\epsilon_3 + 4\epsilon_4 + \epsilon_5.$$

Find a basis for W^0.

Solution. Take f in W^0, and write

$$f(x_1, x_2, x_3, x_4, x_5) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5.$$

Since f annihilates W, we have

$$\begin{aligned} f(\alpha_1) &= c_1 + 2c_2 + c_3 &&= 0 \\ f(\alpha_2) &= \phantom{c_1 + {}} c_2 + 3c_3 + 3c_4 + c_5 &&= 0 \\ f(\alpha_3) &= c_1 + 4c_2 + 6c_3 + 4c_4 + c_5 &&= 0. \end{aligned}$$

The coefficient matrix for this system reduces as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

Since the latter matrix has three nonzero rows, we see that W has dimension 3 and $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for W. We also see that

$$\begin{aligned} c_1 &= -4c_4 - 3c_5, \\ c_2 &= 3c_4 + 2c_5, \end{aligned}$$

and

$$c_3 = -2c_4 - c_5.$$

Therefore the set $\{f_1, f_2\}$, where

$$f_1(x_1, x_2, x_3, x_4, x_5) = -4x_1 + 3x_2 - 2x_3 + x_4$$

and

$$f_2(x_1, x_2, x_3, x_4, x_5) = -3x_1 + 2x_2 - x_3 + x_5,$$

is a basis for W^0. Note that $\dim W^0 = 2$, which agrees with Theorem 16. □

3.5.9 Exercise 9

Let V be the vector space of all 2×2 matrices over the field of real numbers, and let

$$B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Let W be the subspace of V consisting of all A such that $AB = 0$. Let f be a linear functional on V which is in the annihilator of W. Suppose that $f(I) = 0$ and $f(C) = 3$, where I is the 2×2 identity matrix and

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find $f(B)$.

Solution. Note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2a - b & -2a + b \\ 2c - d & -2c + d \end{bmatrix} = 0$$

if and only if $2a = b$ and $2c = d$. Therefore, a basis for W is

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}.$$

Now, let

$$f(A) = c_1A_{11} + c_2A_{12} + c_3A_{21} + c_4A_{22},$$

where A is any 2×2 matrix. We know that f annihilates the two basis vectors for W found above, and we also know that $f(I) = 0$ and $f(C) = 3$. This leads to the following system of linear equations:

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_3 + 2c_4 &= 0 \\ c_1 + c_4 &= 0 \\ c_4 &= 3. \end{aligned}$$

This system has the unique solution

$$c_1 = -3, \quad c_2 = \frac{3}{2}, \quad c_3 = -6, \quad c_4 = 3,$$

so

$$f(A) = -3A_{11} + \frac{3}{2}A_{12} - 6A_{21} + 3A_{22}.$$

Therefore

$$f(B) = -3(2) + \frac{3}{2}(-2) - 6(-1) + 3(1) = 0. \qquad \square$$

3.5.10 Exercise 10

Let F be a subfield of the complex numbers. We define n linear functionals on F^n $(n \geq 2)$ by

$$f_k(x_1, \ldots, x_n) = \sum_{j=1}^{n} (k - j)x_j, \quad 1 \leq k \leq n.$$

What is the dimension of the subspace annihilated by $f_1, \ldots, f_n$?

Solution. Call the subspace W. We first want to find $\dim W^0$. Fix some $n \geq 2$ and consider the set $\mathcal{B} = \{f_1, f_2\}$. Since $f_1(\alpha) = f_2(\alpha) = 0$ for any α in W, we have

$$\begin{aligned} -x_2 - 2x_3 - 3x_4 - \cdots - (n-1)x_n &= 0, \\ x_1 - x_3 - 2x_4 - \cdots - (n-2)x_n &= 0. \end{aligned}$$

The coefficient matrix for the above system of equations is given by

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 & \cdots & 1-n \\ 1 & 0 & -1 & -2 & \cdots & 2-n \end{bmatrix}.$$

Observe that by multiplying the first row by -1 and interchanging the two rows, we can put A in row-reduced echelon form. Since A is row-equivalent to a row-reduced matrix having two nonzero rows, we see that the set $\mathcal{B}$ is linearly independent.

Next, consider f_k for some k with $3 \le k \le n$. Then we can write

$$\begin{aligned}
f_k(x_1, \ldots, x_n) &= \sum_{j=1}^{n} (k-j)x_j \\
&= \sum_{j=1}^{n} (-k - 2j + 2 + jk)x_j + \sum_{j=1}^{n} (2k + j - 2 - jk)x_j \\
&= (2-k)\sum_{j=1}^{n} (1-j)x_j + (k-1)\sum_{j=1}^{n} (2-j)x_j \\
&= (2-k)f_1(x_1, \ldots, x_n) + (k-1)f_2(x_1, \ldots, x_n).
\end{aligned}$$

This shows that $\mathcal{B}$ spans the annihilator of W. Therefore $\mathcal{B}$ is a basis for W^0 and $\dim W^0 = 2$. It now follows from Theorem 16 that $\dim W = n - 2$. □

3.5.11 Exercise 11

Let W_1 and W_2 be subspaces of a finite-dimensional vector space V.

(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

Proof. If a linear functional f belongs to $(W_1 + W_2)^0$, then it annihilates every vector in $W_1 + W_2$. But W_1 and W_2 are subspaces of $W_1 + W_2$, so f must belong to $W_1^0 \cap W_2^0$.

Conversely, if f belongs to $W_1^0 \cap W_2^0$, then f annihilates all vectors in W_1, and also annihilates all vectors in W_2. Since f is linear, it must therefore annihilate sums of these vectors, so f is in $(W_1 + W_2)^0$.

We have shown that members of $(W_1 + W_2)^0$ are members of $W_1^0 \cap W_2^0$ and vice versa, so these spaces are equal. □

(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Proof. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ be a basis for $W_1 \cap W_2$. Extend this basis to a basis $\mathcal{B}'$ for W_1, where

$$\mathcal{B}' = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}.$$

Also extend $\mathcal{B}$ to a basis $\mathcal{B}''$ for W_2, where

$$\mathcal{B}'' = \{\alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_p\}.$$

Now take any linear functional f belonging to $(W_1 \cap W_2)^0$. Let g be the linear functional on V such that

$$\begin{gathered}
g(\alpha_1) = \cdots = g(\alpha_m) = 0, \\
g(\beta_1) = \cdots = g(\beta_n) = 0, \\
g(\gamma_1) = f(\gamma_1), \quad \ldots, \quad g(\gamma_p) = f(\gamma_p).
\end{gathered}$$

Notice that g belongs to W_1^0. Similarly, define h to be the linear functional given by

$$h(\alpha_1) = \cdots = h(\alpha_m) = 0,$$
$$h(\gamma_1) = \cdots = h(\gamma_p) = 0,$$
$$h(\beta_1) = f(\beta_1), \quad \ldots, \quad h(\beta_n) = f(\beta_n).$$

Notice that h belongs to W_2^0. Moreover, $f = g + h$. This shows that f belongs to the sum $W_1^0 + W_2^0$, and we see that $(W_1 \cap W_2)^0$ is a subset of $W_1^0 + W_2^0$.

Next, suppose $f = f_1 + f_2$, where f_1 is in W_1^0 and f_2 is in W_2^0. Since $W_1 \cap W_2$ is a subspace of W_1 and also a subspace of W_2, it follows that both f_1 and f_2 annihilate $W_1 \cap W_2$, i.e. f belongs to $(W_1 \cap W_2)^0$. This completes the proof that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$. □

3.5.12 Exercise 12

Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for each α in the subspace W.

Proof. Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ be a basis for W and extend this basis to a basis

$$\mathcal{B}' = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}$$

for V.

Let f be any linear functional on W. Define the linear functional g on V by

$$g(\alpha_i) = f(\alpha_i), \quad \text{for } 1 \le i \le m, \quad \text{and} \quad g(\beta_j) = 0, \quad \text{for } 1 \le j \le n.$$

We know g exists by Theorem 1. So g is a linear functional on V that agrees with f on W, as we wanted to show. □

3.5.13 Exercise 13

Let F be a subfield of the field of complex numbers and let V be any vector space over F. Suppose that f and g are linear functionals on V such that the function h defined by $h(\alpha) = f(\alpha)g(\alpha)$ is also a linear functional on V. Prove that either $f = 0$ or $g = 0$.

Proof. Let f, g, and h be the linear functionals on V with the properties described above.

Choose any α, β in V. Then

$$\begin{aligned}
h(\alpha + \beta) &= f(\alpha + \beta)g(\alpha + \beta) \\
&= (f(\alpha) + f(\beta))(g(\alpha) + g(\beta)) \\
&= f(\alpha)g(\alpha) + f(\alpha)g(\beta) + f(\beta)g(\alpha) + f(\beta)g(\beta) \\
&= h(\alpha) + h(\beta) + f(\alpha)g(\beta) + f(\beta)g(\alpha) \\
&= h(\alpha + \beta) + f(\alpha)g(\beta) + f(\beta)g(\alpha).
\end{aligned}$$

This shows that for any pair of vectors α, β in V,

$$f(\alpha)g(\beta) + f(\beta)g(\alpha) = 0. \tag{3.3}$$

Taking $\beta = \alpha$, we also see that $h = 0$.

Now, either $f = 0$ or $f \neq 0$. If $f = 0$ then there is nothing left to prove, so we will assume that $f \neq 0$. Again, let α in V be arbitrary, and let β in V be such that $f(\beta)$ is nonzero. We know that

$$f(\beta)g(\beta) = 0,$$

and it follows that $g(\beta) = 0$ (since $f(\beta) \neq 0$). Substituting zero for $g(\beta)$ in equation (3.3) then gives

$$f(\beta)g(\alpha) = 0.$$

But, again, $f(\beta)$ is nonzero, so we must have $g(\alpha) = 0$. Since α was chosen arbitrarily, it follows that $g = 0$ and the proof is complete. □

3.5.14 Exercise 14

Let F be a field of characteristic zero and let V be a finite-dimensional vector space over F. If $\alpha_1, \ldots, \alpha_m$ are finitely many vectors in V, each different from the zero vector, prove that there is a linear functional f on V such that

$$f(\alpha_i) \neq 0, \quad i = 1, \ldots, m.$$

Proof. Let $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ be a basis for V. We will use induction on m to show that we can always find a linear functional f such that $f(\alpha_i) \neq 0$ for all i.

First, when $m = 1$, we have a single nonzero vector α_1. Writing α_1 as a linear combination of the vectors in $\mathcal{B}$ gives

$$\alpha_1 = c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n,$$

for some scalars $c_1, \ldots, c_n$ in F. Since $\alpha_1 \neq 0$, there is an index k such that $c_k \neq 0$. Now define f to be the linear functional on V such that

$$f(\beta_i) = \delta_{ik}c_k^{-1},$$

where δ_{ik} is the Kronecker delta. We now have

$$f(\alpha_1) = c_k c_k^{-1} = 1 \neq 0.$$

This shows that the statement holds for the base case of $m = 1$.

Now assume that the statement is true when $m = k$ for some $k \geq 1$, and let $k + 1$ nonzero vectors $\alpha_1, \ldots, \alpha_{k+1}$ be given. We may apply the inductive hypothesis to find a linear functional f_0 on V such that $f_0(\alpha_i) \neq 0$ for all i with $1 \leq i \leq k$.

If $f_0(\alpha_{k+1})$ happens to be nonzero, then we are done. So we will assume that $f_0(\alpha_{k+1}) = 0$. Again the inductive hypothesis, when applied to the single vector α_{k+1}, allows us to find a linear functional f_1 such that $f_1(\alpha_{k+1}) \neq 0$.

For each i with $1 \leq i \leq k + 1$, define the number N_i as follows. First, if there is no positive integer n such that

$$f_0(\alpha_i) + nf_1(\alpha_i) = 0, \tag{3.4}$$

then set $N_i = 1$. Otherwise, define N_i to be the unique positive integer such that

$$f_0(\alpha_i) + N_i f_1(\alpha_i) = 0.$$

In this second case, we know that N_i is unique for the following reason. Suppose $n = M$ and $n = N_i$ both satisfy (3.4). By construction, it is not possible for $f_0(\alpha_i)$ and $f_1(\alpha_i)$ to both be zero. But if one is zero, then (3.4) implies that the other is as well (because F has characteristic zero). So we see that neither $f_0(\alpha_i)$ nor $f_1(\alpha_i)$ is zero, hence

$$M = -\frac{f_0(\alpha_i)}{f_1(\alpha_i)} = N_i.$$

So N_i is well-defined for all $i = 1, \ldots, k+1$.

Now, the set

$$A = \{N_1, N_2, \ldots, N_{k+1}\}$$

is a finite set of natural numbers, so we can find a largest element in A. Take any natural number P greater than this largest element, and define the linear functional f on V by

$$f = f_0 + P f_1.$$

Then we see that $f(\alpha_i) \neq 0$ for each $i = 1, \ldots, k+1$, completing the inductive step of the proof.

By induction, the original statement must be true for all positive integers m. □

3.5.15 Exercise 15

According to Exercise 3.5.3, similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to be the trace of any matrix which represents the operator in an ordered basis. This is well-defined since all such representing matrices for one operator are similar.

Now let V be the space of all 2×2 matrices over the field F and let P be a fixed 2×2 matrix. Let T be the linear operator on V defined by $T(A) = PA$. Prove that $\operatorname{trace}(T) = 2 \operatorname{trace}(P)$.

Proof. Let

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be an ordered basis for V. We calculate

$$\begin{aligned}
T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

From this, we see that the matrix for T relative to $\mathcal{B}$ is the 4×4 matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

We can now readily see that

$$\operatorname{trace}(T) = \operatorname{tr}([T]_{\mathcal{B}}) = 2a + 2d = 2\operatorname{tr}(P). \qquad \square$$

3.5.16 Exercise 16

Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.

Proof. Let

$$\mathcal{B} = \{\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{1n}, \dots, \epsilon_{n1}, \epsilon_{n2}, \dots, \epsilon_{nn}\}$$

be the basis for W where ϵ_{ij} is the $n \times n$ matrix having a 1 in the i, jth entry and all other entries 0. Since f is linear, we may write it as

$$f(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} A_{ij}, \tag{3.5}$$

where each C_{ij} is a fixed constant and A_{ij} is the i, jth entry of A.

Note that the p, qth entry of ϵ_{ij} is $\delta_{ip}\delta_{qj}$, where δ_{ij} is the Kronecker delta. So, if we fix some indices i, j, a, b, each in the range from 1 to n, then we have

$$f(\epsilon_{ij}\epsilon_{ab}) = \sum_{p=1}^{n} \sum_{q=1}^{n} C_{pq} \sum_{k=1}^{n} (\delta_{ip}\delta_{kj})(\delta_{ak}\delta_{qb}) = C_{ib}\delta_{aj}. \tag{3.6}$$

From this, we see that

$$f(\epsilon_{ij}\epsilon_{ji}) = C_{ii} \quad \text{and} \quad f(\epsilon_{ji}\epsilon_{ij}) = C_{jj}.$$

Since these must be equal, we see that $C_{11} = C_{22} = \cdots = C_{nn}$. On the other hand, (3.6) also gives

$$f(\epsilon_{i1}\epsilon_{1j}) = C_{ij} \quad \text{and} \quad f(\epsilon_{1j}\epsilon_{i1}) = C_{11}\delta_{ij}.$$

These must be equal, so by looking at values of i and j where $i \neq j$, we see that $C_{ij} = 0$ whenever $i \neq j$. Therefore, equation (3.5) can be simplified to

$$f(A) = \sum_{k=1}^{n} C A_{kk} = C(A_{11} + A_{22} + \cdots + A_{nn}) = C\operatorname{tr}(A),$$

where $C = C_{11}$ is a constant. We conclude that f is a scalar multiple of the trace function. If we require $f(I) = n$, then we must have $C = 1$ and $f = \operatorname{tr}$. $\square$

3.5.17 Exercise 17

Let W be the space of $n \times n$ matrices over the field F, and let W_0 be the subspace spanned by the matrices C of the form $C = AB - BA$. Prove that W_0 is exactly the subspace of matrices which have trace zero.

Proof. Let W_1 be the subspace of matrices having trace zero. From Exercise 3.5.3, we know that

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0,$$

so every matrix of the form $AB - BA$ must be in W_1. If we can show that $\dim W_0 = \dim W_1$, then the proof will be complete.

Since W_1 is the null space of a nonzero linear functional, it must have dimension equal to $\dim(W) - 1 = n^2 - 1$. Since W_0 is a proper subset of W, it must have dimension at most $n^2 - 1$. So we need only find a set of $n^2 - 1$ linearly independent matrices in W_0.

Let ϵ_{ij} denote the matrix whose i, jth entry is 1 and all other entries 0. For each i with $2 \leq i \leq n$, let

$$S_i = \epsilon_{ii} - \epsilon_{11} = \epsilon_{i1}\epsilon_{1i} - \epsilon_{1i}\epsilon_{i1}.$$

Then the set $S = \{S_2, \ldots, S_n\}$ is a subset of W_0. Let T be the set of matrices ϵ_{ij} with $i \neq j$. Then T is also a subset of W_0, since we can write

$$\epsilon_{ij} = \epsilon_{ij}\epsilon_{jj} - \epsilon_{jj}\epsilon_{ij}, \quad i \neq j$$

(note that $\epsilon_{jj}\epsilon_{ij} = 0$). Now the set $S \cup T$ consists of

$$(n-1) + (n^2 - n) = n^2 - 1$$

linearly independent vectors belonging to W_0, as needed to complete the proof. $\square$

3.6 The Double Dual

3.6.1 Exercise 1

Let n be a positive integer and F a field. Let W be the set of all vectors $(x_1, \dots, x_n)$ in F^n such that $x_1 + \cdots + x_n = 0$.

(a) Prove that W^0 consists of all linear functionals f of the form

$$f(x_1, \dots, x_n) = c \sum_{j=1}^{n} x_j.$$

Proof. If $n = 1$, then W is the zero subspace and the result is trivial, so we will suppose $n > 1$.

Let f be in W^0. We can find scalars $c_1, \dots, c_n$ in F such that

$$f(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n.$$

We want to show that $c_1 = c_2 = \cdots = c_n$.

Since f annihilates W, in particular we know that

$$f(1, -1, 0, 0, \dots, 0) = c_1 - c_2 = 0.$$

Therefore $c_1 = c_2$. Likewise, we know that

$$f(0, 1, -1, 0, 0, \dots, 0) = c_2 - c_3 = 0,$$

so $c_2 = c_3$. Continuing in this way, we see that the c_i must all be identical. Thus f has the form that was specified. □

(b) Show that the dual space W^* of W can be 'naturally' identified with the linear functionals

$$f(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

on F^n which satisfy $c_1 + \cdots + c_n = 0$.

Proof. Let U be the space of linear functionals

$$f(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

on F^n which satisfy $c_1 + \cdots + c_n = 0$. We will show that U is isomorphic to W^*.

Let T be the function from U into W^* such that $T(f)$ is the restriction of f to W. Then T is a linear transformation. We will show that it is non-singular. Suppose $T(f) = 0$. Then $f(\alpha) = 0$ for all α in W, hence f belongs to W^0. By the result from part (a), we know that f has the form

$$f(x_1, \dots, x_n) = c \sum_{j=1}^{n} x_j.$$

But f belongs to U, so c must be 0 and f is the zero functional. That is, we have shown that $T(f) = 0$ implies $f = 0$, so T is non-singular and hence one-to-one.

Now, it can be shown that U and W both have dimension $n - 1$. For example, if $\{\epsilon_1, \ldots, \epsilon_n\}$ is the standard ordered basis for F^n, then

$$\{\epsilon_1 - \epsilon_i \mid 2 \leq i \leq n\}$$

is a set of $n - 1$ linearly independent vectors which span W. We can find a similar basis for U. Since T is a one-to-one linear transformation between vector spaces of equal dimension, T must be invertible and thus is an isomorphism. $\square$

3.7 The Transpose of a Linear Transformation

3.7.1 Exercise 1

Let F be a field and let f be the linear functional on F^2 defined by

$$f(x_1, x_2) = ax_1 + bx_2.$$

For each of the following linear operators T, let $g = T^t f$, and find $g(x_1, x_2)$.

(a) $T(x_1, x_2) = (x_1, 0)$

Solution. We have

$$\begin{aligned} g(x_1, x_2) &= (T^t f)(x_1, x_2) \\ &= f(T(x_1, x_2)) \\ &= f(x_1, 0) \\ &= ax_1. \end{aligned}$$ □

(b) $T(x_1, x_2) = (-x_2, x_1)$

Solution. In this case, we get

$$\begin{aligned} g(x_1, x_2) &= f(T(x_1, x_2)) \\ &= f(-x_2, x_1) \\ &= -ax_2 + bx_1. \end{aligned}$$ □

(c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$

Solution.

$$\begin{aligned} g(x_1, x_2) &= f(x_1 - x_2, x_1 + x_2) \\ &= (b + a)x_1 + (b - a)x_2. \end{aligned}$$ □

3.7.2 Exercise 2

Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by

$$f(p) = \int_a^b p(x)\, dx.$$

If D is the differentiation operator on V, what is $D^t f$?

Solution. From the definition, we have

$$(D^t f)(p) = f(Dp) = \int_a^b (Dp)(x)\, dx.$$

So, the fundamental theorem of calculus gives

$$(D^t f)(p) = p(b) - p(a).$$ □

3.7.3 Exercise 3

Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is the linear operator on V defined by $T(A) = AB - BA$, and if f is the trace function, what is $T^t f$?

Solution. From Exercise 3.5.3, we know that $f(AB) = f(BA)$. So,

$$\begin{aligned}(T^t f)(A) &= f(TA) \\ &= f(AB - BA) \\ &= f(AB) - f(BA) \\ &= 0,\end{aligned}$$

and we see that $T^t f = 0$. □

3.7.4 Exercise 4

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Let c be a scalar and suppose there is a non-zero vector α in V such that $T\alpha = c\alpha$. Prove that there is a non-zero linear functional f on V such that $T^t f = cf$.

Proof. Let $n = \dim V$ and let α be a nonzero vector in V with $T\alpha = c\alpha$. Define the linear operator U on V by

$$U = T - cI.$$

Then $U\alpha = T\alpha - c\alpha = 0$. Therefore α belongs to the null space of U, which implies (by Theorem 2) that $\operatorname{rank}(U) < n$. Now consider the linear operator U^t on V^*. By Theorem 22, we have

$$\operatorname{rank}(U^t) = \operatorname{rank}(U) < n,$$

so the nullspace of U^t has dimension greater than zero. Therefore we can find a nonzero linear functional f in V^* such that $U^t f = 0$. Then for any β in V,

$$\begin{aligned}0 &= (U^t f)(\beta) \\ &= f(U\beta) \\ &= f(T\beta - c\beta) \\ &= f(T\beta) - cf(\beta) \\ &= (T^t f)(\beta) - cf(\beta).\end{aligned}$$

So, we have $T^t f = cf$ as required. □

3.7.5 Exercise 5

Let A be an $m \times n$ matrix with *real* entries. Prove that $A = 0$ if and only if $\operatorname{trace}(A^t A) = 0$.

Proof. Certainly if $A = 0$, then $A^t A = 0$ and $\text{trace}(A^t A) = 0$. We now need only prove the converse. Let $B = A^t$ and suppose $\text{trace}(BA) = 0$. Then

$$0 = \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2.$$

Since we have a sum of squares (of real numbers) equal to zero, it must be the case that each squared number is zero. In particular $A_{ij} = 0$ for each i, j, since every such entry appears in the sum. □

3.7.6 Exercise 6

Let n be a positive integer and let V be the space of all polynomial functions over the field of real numbers which have degree at most n, i.e., functions of the form

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Let D be the differentiation operator on V. Find a basis for the null space of the transpose operator D^t.

Solution. Let W denote the range of D. By Theorem 22, the null space of D^t is the annihilator of W. We know that W is the space of polynomials having degree at most $n - 1$, so $\dim W = n - 1$. By Theorem 16, the annihilator W^0 must have dimension $n - (n - 1) = 1$, so we may take any nonzero linear functional in W^0 as a basis vector. Let g be the unique linear functional such that

$$g(x^k) = \delta_{nk}, \quad 1 \le k \le n.$$

That is, g sends every polynomial to the coefficient for its x^n term. In particular, g annihilates W, so $\{g\}$ is a basis for W^0 and therefore also a basis for the null space of D^t. □

3.7.7 Exercise 7

Let V be a finite-dimensional vector space over the field F. Show that $T \to T^t$ is an isomorphism of $L(V, V)$ onto $L(V^*, V^*)$.

Proof. Suppose V has dimension n and let U be the function from $L(V, V)$ into $L(V^*, V^*)$ given by $U(T) = T^t$. We could show that U is an isomorphism by appealing to the definition of the transpose. Instead, we will write U as a composition of three linear transformations U_1, U_2, and U_3, mapping

$$L(V, V) \xrightarrow{U_1} F^{n \times n} \xrightarrow{U_2} F^{n \times n} \xrightarrow{U_3} L(V^*, V^*),$$

as follows. U_1 sends an operator T to its matrix $[T]_{\mathcal{B}}$ in some fixed basis $\mathcal{B}$, U_2 sends an $n \times n$ matrix A to its transpose A^t, and U_3 sends an $n \times n$ matrix $[T']_{\mathcal{B}^*}$ to its corresponding linear operator T' on V^*. Then $U = U_3 U_2 U_1$ and it follows that U is a linear transformation. Moreover, Theorem 12 shows that U_1 and U_3 are isomorphisms, and U_2 is an isomorphism since it is obviously invertible (it is its own inverse). Therefore U is an isomorphism from $L(V, V)$ onto $L(V^*, V^*)$. □

3.7.8 Exercise 8

Let V be the vector space of $n \times n$ matrices over the field F.

(a) If B is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(A) = \operatorname{trace}(B^t A)$. Show that f_B is a linear functional on V.

Proof. Since the trace function is linear, we have

$$\begin{aligned} f_B(A_1 + cA_2) &= \operatorname{trace}(B^t(A_1 + cA_2)) \\ &= \operatorname{trace}(B^t A_1 + cB^t A_2) \\ &= \operatorname{trace}(B^t A_1) + c \operatorname{trace}(B^t A_2) \\ &= f_B(A_1) + c f_B(A_2). \end{aligned}$$

Therefore f_B is linear. □

(b) Show that every linear functional on V is of the above form, i.e., is f_B for some B.

Proof. Let g be a linear functional on V. Then g can be written in the form

$$g(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} A_{ij},$$

where each c_{ij} is a fixed scalar in F. Now let B be the matrix whose i, j entry is c_{ij}. Then for each matrix A in V,

$$\begin{aligned} f_B(A) = \operatorname{trace}(B^t A) &= \sum_{j=1}^{n} (B^t A)_{jj} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} (B^t)_{ji} A_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} A_{ij} = g(A). \end{aligned}$$

Therefore $g = f_B$. □

(c) Show that $B \to f_B$ is an isomorphism of V onto V^*.

Proof. Let T denote the function $B \to f_B$. Then

$$\begin{aligned} T(A_1 + cA_2)(A) &= \operatorname{trace}((A_1 + cA_2)^t A) \\ &= \operatorname{trace}(A_1^t A + cA_2^t A) \\ &= \operatorname{trace}(A_1^t A) + c \operatorname{trace}(A_2^t A) \\ &= f_{A_1}(A) + c f_{A_2}(A) \\ &= T(A_1)(A) + cT(A_2)(A), \end{aligned}$$

so T is a linear transformation. Since we have already proven that every linear functional on V can be written as f_B for some matrix B, it follows that T is onto. And since $\dim V = \dim V^* = n^2$, this is enough to show that T is an isomorphism (by Theorem 9). □